HW 1 MATH 170B. (1)

$\boxed{1.1.5}$ (a). Since $f^{(n)}(x) = (-1)^{n-1}(n-1)!(x+1)^{-n}$ with $f(0)=0$

$$\ln(x+1) = \sum_{i=1}^{n} \frac{(-1)^{i-1}(i-1)!\,(0+1)^{-i}}{i!}(x-0)^{i} + R_n(x)$$

$$= \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} x^{i} + R_n(x),$$

where $R_n(x) = \frac{1}{(n+1)!}(-1)^n n!\,(\xi+1)^{-(n+1)}(x-0)^{n+1}$

$$= \frac{(-1)^n}{n+1}(\xi+1)^{-(n+1)} x^{n+1}, \quad \xi \in [0, x]$$

or $R_n(x) = \frac{1}{n!}\int_0^x \frac{(-1)^n}{n+1}(t+1)^{-(n+1)}(x-t)^n\,dt$.

(b) For $\ln 1.5$, take $x=0.5$,

One has, the error is actually remainder

$$\text{Error} = |f(0.5) - T_n(0.5)| = |R_n(0.5)|$$

Consider $|R_n(0.5)| = \left|\frac{(-1)^n}{n+1}(\xi+1)^{-(n+1)}0.5^{n+1}\right| \le \frac{0.5^{n+1}}{n+1}$, since $\xi \ge 0$.

Then find the minimum n s.t. $\frac{0.5^{n+1}}{n+1} \le 10^{-8}$

this can be found by a simple matlab code.

(c) Similarly as part (b), only need to find the minimum n

s.t. $\frac{0.6^{n+1}}{n+1} \le 10^{-10}$

$\boxed{1.1.10}$ This statement is True.

Proof: Since f is differentiable at x,

i.e. $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists.

Now, we only need to show $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{f(x+\bar{h}) - f(x+\alpha \bar{h})}{\bar{h} - \alpha \bar{h}}$, $\alpha \neq 1$

~~Case 1: $\alpha \neq 0$~~

Consider $\frac{f(x+h) - f(x+\alpha h)}{h - \alpha h} = \frac{f(x+h) - f(x)}{(1-\alpha) h} + \frac{\alpha}{1-\alpha} \frac{f(x) - f(x+\alpha h)}{\alpha h}$

taking limits on both sides, we have

$$\lim_{h \to 0} \frac{f(x+h) - f(x+\alpha h)}{h - \alpha h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(1-\alpha) h} + \lim_{h \to 0} \frac{\alpha}{1-\alpha} \frac{f(x) - f(x+\alpha h)}{\alpha h}$$

$$= \frac{1}{1-\alpha} f'(x) - \frac{\alpha}{1-\alpha} f'(x) = f'(x)$$

$\square$

$\boxed{1.1.26}$ Similarly as 1.1.5 parts (a) and (b).

$\boxed{1.2.7}$ (a) $\frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2}$

Since $\lim_{n\to\infty} \frac{n+1}{n^2} / \frac{1}{n} \neq 0$, $\frac{n+1}{n^2} \neq o(\frac{1}{n})$ X

(b) $\frac{n+1}{\sqrt{n}} = \sqrt{n} + \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{n+1}{\sqrt{n}} / 1 \neq 0$, $\frac{n+1}{\sqrt{n}} \neq o(1)$ X

(c) Since $\frac{n}{\ln n} \to \infty \Rightarrow$ For any $C>0$, there always $\exists N$, s.t. $\frac{n}{\ln n} > C$, $n>N$

i.e. $\frac{1}{\ln n} > \frac{C}{n}$, $n>N$.

$\Rightarrow \frac{1}{\ln n} \neq O(\frac{1}{n})$ X

(d) $\frac{1}{n\ln n} / \frac{1}{n} = \frac{1}{\ln n} \to 0$, thus $\frac{1}{n\ln n} = o(1/n)$ ✓

(e) $\frac{e^n}{n^5} \to \infty \Rightarrow \frac{e^n}{n^5} \neq O(1/n)$. X

$\boxed{1.2.15}$ (a) Show $\forall x \in \mathbb{R}$, $\exists$ a $y \in \mathbb{R}$ s.t. $x = y - \varepsilon \sin y$

Since $y - \varepsilon \sin y \to +\infty$ if $y \to +\infty$

$y - \varepsilon \sin y \to -\infty$ if $y \to -\infty$

By ~~I~~VT, since $y - \varepsilon \sin y$ is continuous, for any $x \in \mathbb{R}$, there is always a y s.t. $y - \varepsilon \sin y = x$.

$\boxed{1.2.15}$ (b) $x = y - \varepsilon \sin y$

taking d/dx on both sides,

$$1 = \frac{dy}{dx} - \varepsilon \cos y \frac{dy}{dx} = \frac{dy}{dx}(1 - \varepsilon \cos(y))$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\varepsilon \cos y}, \quad \varepsilon \in [0,1). \Rightarrow 1 - \varepsilon \cos y \neq 0.$$

thus $\frac{dy}{dx}$ is cont everywhere.

$\boxed{1.2.22}$ proof: Assume $x_n = \frac{1+a\theta^n}{1+a\theta^{n-1}} \Rightarrow x_n - 1 = \frac{a\theta^{n-1}(\theta-1)}{1+a\theta^{n-1}}$

then we have $x_{n+1} = \frac{1+a\theta^{n+1}}{1+a\theta^n} \Rightarrow x_{n+1} - 1 = \frac{a\theta^n(\theta-1)}{1+a\theta^n}$

$$\Rightarrow \frac{|x_{n+1}-1|}{|x_n - 1|} = \left|\frac{\theta + a\theta^n}{1+a\theta^n}\right| \le C < 1, \quad \forall n > N$$

where $|a\theta^N| < \frac{1-\theta}{2}$.

Hence, $x_n \to 1$ linearly. (for $a<0$, pick different N). $\square$

$\boxed{1.2.33}$ proof: Since $x_n = O(\alpha_n) \Leftrightarrow |x_n| \le C|\alpha_n|, \forall n \ge N$

i.e. $\frac{|x_n|}{|\alpha_n|} \le C, \quad \forall n \ge N$

Look at $\frac{x_n}{\ln n \, \alpha_n} = \frac{1}{\ln n} \cdot \frac{x_n}{\alpha_n}$

$$\Rightarrow -\left|\frac{x_n}{\alpha_n}\right| \frac{1}{\ln n} \le \frac{x_n}{\ln n \alpha_n} \le \frac{1}{\ln n}\left|\frac{x_n}{\alpha_n}\right|$$

$$\Rightarrow -\frac{C}{\ln n} \le \frac{x_n}{\ln n \alpha_n} \le \frac{C}{\ln n}, \quad \forall n \ge N$$

$$\Rightarrow \frac{x_n}{\ln n \alpha_n} \to 0 \Rightarrow \frac{x_n}{\ln n} = o(\alpha_n). \quad \square$$

1.2.35 Proof:

(i) if $x_n = o(\alpha_n) \Rightarrow x_n = O(\alpha_n)$

this is obviously, if $\frac{x_n}{\alpha_n} \to 0$

there must be a $C > 0$ and a N.

$$\text{s.t. } \left| \frac{x_n}{\alpha_n} \right| < C, \quad \forall n \geq N$$

(ii) if $x_n = O(\alpha_n) \not\Rightarrow x_n = o(\alpha_n)$.

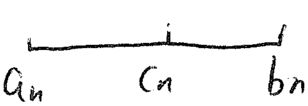
The conterexample is $x_n = O(x_n)$

but $x_n \neq o(x_n)$

3.1.2 proof:

(a). The width of interval at nth step is $(b_0 - a_0) 2^{-n}$.

(b).



, the root r is at most at the boundary a_n or b_n

$\Rightarrow$ Distance $\leq (b_0 - a_0) 2^{-(n+1)}$.

$\boxed{3.1.4}$ proof: By (b) of 3.1.2

we know $|r-c_n| \leq (b_0-a_0)2^{-(n+1)}$

Thus, if we can find an n

s.t. $(b_0-a_0)2^{-(n+1)} \leq \varepsilon$

~~it must~~ with this n, $|r-c_n| \leq \varepsilon$.

So, we only need to solve ineq

$(b_0-a_0)2^{-(n+1)} \leq \varepsilon$ for n.

$$\Rightarrow \quad 2^{-(n+1)} \leq \frac{\varepsilon}{b_0-a_0}$$

$$\Rightarrow \quad -(n+1) \leq \frac{\log \varepsilon - \log(b_0-a_0)}{\log 2}$$

$$n \geq \frac{\log(b_0-a_0) - \log \varepsilon}{\log 2} - 1.$$

$\square$