MATH 170B HW2

3.2.5

Consider $X_{n+1} = 2X_n - X_n^2 y$

as $X_{n+1} = X_n - (X_n^2 y - X_n)$

Assume $X_n^2 y - X_n = \frac{f(X_n)}{f'(X_n)}$ with $f(X) = e^{g(x)}$

thus $\frac{f(x)}{f'(x)} = \frac{e^{g(x)}}{e^{g(x)} \cdot g'(x)} = \frac{1}{g'(x)} = x^2 y - X$

$\Rightarrow \quad g'(x) = \frac{1}{x^2 y - x} \Rightarrow g(x) = \ln(xy-1) - \ln(x)$

$\Rightarrow f(x) = e^{g(x)} = y - \frac{1}{x}$

$\Rightarrow X_{n+1} = 2X_n - X_n^2 y$ is a Newton iteration for $f(x)=0$ with $f(x) = y - \frac{1}{X}$.

3.2.9 $f(x) = x^3 - 2$, $x_0 = 1$

$\Rightarrow f'(x) = 3x^2$

The Newton's iteration is

$$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)} = X_n - \frac{x_n^3 - 2}{3x_n^2}, \quad n = 0, 1, \cdots$$

$\Rightarrow X_1 = \frac{4}{3}$

$\Rightarrow \boxed{X_2 = \frac{91}{72}} \approx 1.2639$

3.2.13 Consider the Newton's iteration of

$$f(x) = x^2 - 1$$

$$X_n = X_{n-1} - \frac{f(X_{n-1})}{f'(X_{n-1})}$$

$$= X_{n-1} - \frac{X_{n-1}^2 - 1}{2X_{n-1}}$$

$$= \frac{1}{2}X_{n-1} + \frac{1}{2X_{n-1}},$$ notice that if $X_{n-1} \geq 1$ then X_n must be ≥ 1 as well.

$$\leq \frac{1}{2}X_{n-1} + \frac{1}{2}$$

$$\leq \frac{1}{2}\left(\frac{1}{2}X_{n-2} + \frac{1}{2}\right) + \frac{1}{2}$$

$$= \left(\frac{1}{2}\right)^2 X_{n-2} + \left[\frac{1}{2} + \left(\frac{1}{2}\right)^2\right]$$

$$\ldots$$

$$\leq \left(\frac{1}{2}\right)^n X_0 + \left[\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n\right]$$

$$\leq \left(\frac{1}{2}\right)^n X_0 + 1$$

$$\Rightarrow e_n \leq \left(\frac{1}{2}\right)^n X_0 \quad \text{where } e_n = |X_n - 1|$$

thus, just need to find n

s.t. $\left(\frac{1}{2}\right)^n X_0 < 10^{-8}$.

$\boxed{3.2.22}$

First, define $F(x)$ as

$$\begin{cases} f_1(x) = xy - z^2 - 1 \\ f_2(x) = xyz - x^2 + y^2 - 2 \\ f_3(x) = e^x - e^y + z - 3 \end{cases}$$

To find x s.t. $F(x) = 0$,

apply Newton's method,

with $$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} y & x & -2z \\ yz - 2x & xz + 2y & xy \\ e^x & -e^y & 1 \end{bmatrix}$$

and $x_0 = (0, 0, 1)$

Solve $J(x_0) h = -F(x_0)$ for h

$\Rightarrow$ ~~$x_1 = x_0 - F(x_0)/F'(x_0)$~~

Since $J(x_0)$ is singular and $F(x_0) \notin \text{Range}(J(x_0))$ there is no solution h.

~~$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$~~

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3.3.4 $f(x) = x^2 - 2$, $x_0 = 0$, $x_1 = 1$ (4)

the secant iteration of $f(x) = 0$ is

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]$$

$$\Rightarrow x_2 = x_1 - f(x_1)\left[\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right]$$

$$= 1 - f(1)\left[\frac{1-0}{f(1) - f(0)}\right]$$

$$= 2.$$

3.3.7

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]$$

$$= \frac{x_n f(x_n) - x_n f(x_{n-1}) - x_n f(x_n) + x_{n-1} f(x_n)}{f(x_n) - f(x_{n-1})}$$

$$= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

The reason why we prefer equ (3) in practice is we can keep using $h_{n+1} \triangleq x_{n+1} - x_n = -f(x_n)\left[\frac{h_n}{f(x_n) - f(x_{n-1})}\right]$ in the next iterations to update $x_{n+2} = x_{n+1} + h_{n+2} \cdots$

3.4.3

Proof: Use Contradiction

Assume there is no fix-point.

Then Either $F(x) - x > 0$ on $[a, b]$ ①

or $F(x) - x < 0$ on $[a,b]$ ②.

For ①, Pick $x = b$, $F(b) - b > 0$

$F(b) > b$

contradiction with $F: [a,b] \to [a,b]$.

For ②, pick $x = a$, $F(a) - a < 0$

$F(a) < a$.

Contradiction with $F: [a, b] \to [a, b]$.

Therefore, F has a fixed-pt.

If $f: \mathbb{R} \to \mathbb{R}$, this statement is not true.

Counterexample: Let $f(x) = x+1$

then $f(x) - x = 1 \neq 0$ on $\mathbb{R}$.

D.

$\boxed{3.4.4}$ (6)

(b) $F(x) = \frac{1}{2}x$

$\Rightarrow |F(x) - F(y)| \le |\frac{1}{2}x - \frac{1}{2}y| \le \frac{1}{2}|x-y|$ with $\lambda = \frac{1}{2}$

(d) $|x|^{\frac{3}{2}}$ on $|x| \le \frac{1}{3} \Leftrightarrow x^{\frac{3}{2}}$ on $0 \le x \le \frac{1}{3}$

By MVT

one has $\left| |x|^{\frac{3}{2}} - |y|^{\frac{3}{2}} \right| \le |x^{\frac{3}{2}} - y^{\frac{3}{2}}|$

$= (x^{\frac{3}{2}})'(\xi)\, |x-y|, \quad \xi \in [0, \frac{1}{3}]$

$= \frac{3}{2}\xi^{\frac{1}{2}} |x-y|$

$\le \frac{3}{2}(\frac{1}{3})^{\frac{1}{2}} |x-y|$ with $\lambda = \frac{3}{2}(\frac{1}{3})^{\frac{1}{2}}$

$\boxed{3.4.12}$ Let $x_{n+1} = \sqrt{p + x_n}$, with $x_1 = \sqrt{p}$.

thus $\lim_{n\to\infty} x_n = \sqrt{p + \sqrt{p + \cdots}} = x$

$\Rightarrow x = \sqrt{p + \sqrt{p + \cdots}} = \sqrt{p + x}$

$\Rightarrow$ x is the positive root

$x = \frac{1 + \sqrt{1+4p}}{2}$

Another way to think:

def $x_1 = \sqrt{p}$

$x_{n+1} = \sqrt{p + x_n}$

This is a fix-pt iteration of $f(x) = \sqrt{p+x}$

and clearly $\lim_{n\to\infty} x_n = \sqrt{p+\sqrt{p+\cdots}}$

Thus, only need to find the fix-pt of $f(x)$, i.e. solve $f(x) = x$ for x.