According to the definition on page 349, To verify \( f(x) \) is a 2nd order spline or not, we only to to check \( \begin{cases} S_0(1) = S_1(1) \\ S_1(2) = S_2(2) \end{cases} \) \( \begin{cases} S_0'(1) = S_1'(1) \\ S_1'(2) = S_2'(2) \end{cases} \), where \( S_0(1) = 1 = S_1(1) \), \( S_0'(1) = 10 = S_1'(1) \) \( S_1(2) = \frac{3}{2} = S_2(2) \), \( S_1'(2) = 0 = S_2'(2) \).

Hence, \( f(x) \) is a quadratic spline.

No, \( f(x) \) is not a cubic spline. The reason is \( S_0''(1) \neq S_1''(1) \).

Totally, we have 5 unknowns, then we need 5 eqs.

In order to be a cubic spline, we need \( \begin{cases} S_0(1) = S_1(1) \\ S_1(3) = S_2(3) \end{cases} \) \( \begin{cases} S_0'(1) = S_1'(1) \\ S_1'(3) = S_2'(3) \end{cases} \) and \( S_0''(1) = S_1''(1) \), \( S_1''(3) = S_2''(3) \).

These give us \( \begin{cases} a = c \\ c = d \end{cases} \)

By the table, we can find \( \begin{cases} 4a + b = 26 \\ c = 7 \\ 4d + e = 25 \end{cases} \) Solve \( \begin{cases} 4a + b = 26 \\ c = 7 \\ 4d + e = 25 \end{cases} \) \( \begin{cases} a = c = d = 7 \\ b = -2 \\ e = -3 \end{cases} \)
For 1st degree spline, it is just the piece-wise linear interpolation, which is

\[
\text{area} = \frac{1}{2} \sum_{i=0}^{n-1} (f(t_{i+1}) + f(t_i))(t_{i+1} - t_i)
\]

Then, the \( \int s(x) \, dx \) is the area of those trapezoids.

Hence, the area can be calculated as

\[
\int s(x) \, dx = \frac{1}{2} \sum_{i=0}^{n-1} (f(t_{i+1}) + f(t_i))(t_{i+1} - t_i)
\]
According to Eq (1) on Page 367,
\[
B_j^2(t_i) = \left( \frac{t_i - t_j}{t_{j+2} - t_j} \right) B_j'(t_i) + \left( \frac{t_{j+3} - t_i}{t_{j+3} - t_{j+1}} \right) B_{j+1}'(t_i)
\]

By the recursive formula of $B_j'(x)$ given on the bottom of Page 367, we have
\[
B_j'(x) = \begin{cases} 
0, & x < t_j \text{ or } x > t_{j+2} \\
\frac{x - t_j}{t_{j+1} - t_j}, & t_j \leq x \leq t_{j+1} \\
\frac{t_{j+2} - x}{t_{j+2} - t_{j+1}}, & t_{j+1} \leq x < t_{j+2}
\end{cases}
\]

Thus, $B_j'(ti) \neq 0$ only if $t_i = t_{j+1}$, then it equals
\[
\left( \frac{t_i - t_j}{t_{j+2} - t_j} \right) B_j'(ti) = \left( \frac{t_{j+1} - t_i}{t_{j+1} - t_{j+2}} \right) \delta_{i,j+1}
\]

which is actually $\left( \frac{t_i - t_{j+1}}{t_{j+1} - t_{j+2}} \right) \delta_{i,j+1}$

For the second term,
\[
\left( \frac{t_{j+3} - t_i}{t_{j+3} - t_{j+1}} \right) B_{j+1}'(t_i) \neq 0, \text{ only if } t_i = t_{j+2},
\]

then, it equals to $\left( \frac{t_{j+3} - t_{j+2}}{t_{j+3} - t_{j+1}} \right)$

which is actually $\left( \frac{t_{j+1} - t_i}{t_{j+1} - t_{j+2}} \right) \delta_{i,j+2}$.
By Lemma 1 on page 368,

that is, if \( k \geq 1 \) and \( x \notin \left( \frac{t_i}{i}, \frac{t_{i+1}}{i+1} \right) \), then \( B_i^k(x) = 0 \).

Case I: When \( k = 0 \), obviously \( \sum_{i=-\infty}^{\infty} C_i B_i^0(x) = \sum_{i=m-k}^{m} C_i B_i^0(x) \).

Case II: When \( k \geq 1 \), since \( x \in [t_m, t_{m+1}) \), by Lemma 1, only terms from \( B_{m-k}^k(x) \) to \( B_m^k(x) \) are non-zero.

Thus, \( \sum_{i=-\infty}^{\infty} C_i B_i^k(x) = \sum_{i=m-k}^{m} C_i B_i^k(x) \).

By 6.5.2, we have

\[
S(t_j) = \sum_{i=-\infty}^{\infty} C_i B_i^2(t_j) = \sum_{i=j-2}^{j} C_i B_i^2(t_j)
\]

By 6.5.1, we have \( B_j^2(t_j) = 0 \), and

\[
S(t_j) = C_{j-2} \left( \frac{t_{j+1} - t_j}{t_{j+1} + t_{j-1}} \right) + C_{j-1} \left( \frac{t_j - t_{j-1}}{t_{j+1} - t_{j-1}} \right)
\]

\[
= C_{j-2} \frac{h_j}{h_j + h_{j-1}} + C_{j-1} \frac{h_{j-1}}{h_j + h_{j-1}}
\]

\[
= y_j
\]
6.5.4 Show by Induction.

Step 1: Obviously, for \( n=1 \), \( B_i^0(x) = B_0^0(x-t_i) \), by the definition of \( B_i^0(x) \).

Step 2: Assume \( k=n \), \( B_i^n(x) = B_0^n(x-t_i) \), then, let's look at the case with \( k=n+1 \).

Simply apply Eq (1) on page 367.

\[
B_i^{n+1}(x) = \left( \frac{x-t_i}{t_{i+k} - t_i} \right) B_i^n(x) + \left( \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} \right) B_{i+1}^n(x).
\]

By Assumption at \( k=n \), we have

\[
B_i^{n+1}(x) = \left( \frac{x-t_i}{t_{i+k} - t_i} \right) B_0^n(x-t_i) + \left( \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} \right) B_1^n(x-t_i)
\]

\[
= \left( \frac{(x-t_i) - t_0}{t_h - t_0} \right) B_0^n(x-t_i) + \left( \frac{t_{k+1} - x + t_i}{t_{k+1} - t_i} \right) B_1^n(x-t_i)
\]

\[
= B_0^{n+1}(x-t_i)
\]

replace every \( k \) by \( n+1 \).

Notice \( t_2 = t_i \).
We just look at each row.

Since $B^k_j(x_j) \neq 0$ for all $j$

$\iff$ by lemma 1, $t_j < x_j < t_{j+k+1}$
$\quad t_{j+1} < x_{j+1} < t_{j+k+2}$
$\quad \vdots$
$\quad t_{j+k} < x_{j+k} < t_{j+k+1}$

Also $t_{j-1} < x_{j-1} < t_{j+k}$
$\quad \vdots$
$\quad t_{j+k} < x_{j-k} < t_{j+1}$

Notice that all points from $x_{j-k}$ to $x_{j+k}$

may be in the interval $(t_j, t_{j+k+1})$

this implies (again by lemma 1)

from $B^k_j(x_{j-k})$ to $B^k_j(x_{j+k})$ may be non-zero,

totally $2k+1$