MATH 170B HW5

$\boxed{7.1.6}$ (a) $f'(x) \approx \frac{1}{12h}[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)]$.

By Taylor's expansion,

$$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(x) + o(h^5)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + o(h^5)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + o(h^5)$$

$$f(x-2h) = f(x) - 2hf'(x) + \frac{4h^2}{2!}f''(x) - \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(x) + o(h^5).$$

$$\Rightarrow \quad -f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h) = 12hf'(x) + o(h^5)$$

$$\Rightarrow f'(x) = \frac{1}{12h}[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)].$$

~~(b)~~ $\boxed{7.1.7}$ Similarly as 7.1.6.

$\boxed{7.1.12}$ $L = \varphi(h) + a_1 h + a_3 h^3 + a_5 h^5$, ... (i)

the purpose of Richardson extrapolation is to eliminate leading error term $a_1 h$.

thus, let $h = \frac{h}{2}$

$$\Rightarrow L = \varphi(\tfrac{h}{2}) + \frac{a_1 h}{2} + \frac{a_3 h^3}{8} + \frac{a_5 h^5}{32}$$

$$\Rightarrow 2L = 2\varphi(\tfrac{h}{2}) + a_1 h + \frac{a_3 h^3}{4} + \frac{a_5 h^5}{16} \quad \text{.....(ii)}$$

(ii) − (i): $L = \left[2\varphi(\tfrac{h}{2}) - \varphi(h)\right] + \left(-\frac{3}{4}a_3 h^3 - \frac{15}{16}a_5 h^5\right)$.

$2\varphi(\tfrac{h}{2}) - \varphi(h)$ is a approximation of L with leading error term $O(h^3)$

$\boxed{7.1.15}$ $f'(x) = \varphi(h) - \frac{h^2}{6}f'''(x) - \frac{h^4}{120}f^{(5)}(x)$ ...... (i)

where $\varphi(h) = \frac{1}{2h}\left[f(x+h) - f(x-h)\right]$.

Let $h = \frac{h}{2}$

$$f'(x) = \varphi(\tfrac{h}{2}) - \frac{h^2}{24}f'''(x) - \frac{h^4}{2^4 \cdot 120}f^{(5)}(x) \quad \text{.....(ii)}$$

4(ii) − (i): $3f'(x) = 4\varphi(\tfrac{h}{2}) - \varphi(h) - \frac{4h^4}{2^4 \cdot 120}f^{(5)}(x) + \frac{h^4}{120}f^{(5)}(x)$

$\Rightarrow$ ~~the~~ $f'(x) = \frac{4}{3}\varphi(\tfrac{h}{2}) - \frac{1}{3}\varphi(h) + O(h^4)$.

7.2.1 By Eq(3) on page 480

The Newton-cotes formula is

$$\int_0^1 f(x)dx \approx \sum_{i=0}^{3} A_i f(x_i),$$

where $A_i = \int_0^1 l_i(x)dx$

where $l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{3} \frac{(x-x_j)}{(x_i-x_j)}$, with $x_0=0$, $x_1=\frac{1}{3}$, $x_2=\frac{2}{3}$, $x_3=1$.

7.2.4 Just compare the given formula with Newton-cotes formula with nodes $x_0=0$, $x_1=\frac{1}{4}$, $x_2=\frac{1}{2}$, $x_3=\frac{3}{4}$, $x_4=1$, if they are identical, then it implies the given ~~formular~~ formula is exact with respect to any degree ≤ 4 polynomials.

$\boxed{7.2.5}$ By subsitibution,

Let $x = y(b-a)+a$

$$\Rightarrow \int_a^b f(x)\,dx = \int_0^1 f(y(b-a)+a)\cdot\frac{dx}{dy}\cdot dy$$

$$= (b-a)\int_0^1 g(y)\,dy, \quad g(y) = f(y(b-a)+a).$$

Since for any dergee ≤ 1 polynomials,

$$\int_0^1 g(y)\,dy = \frac{1}{90}\left[7g(0) + 32g(\tfrac{1}{4}) + 12g(\tfrac{1}{2}) + 32g(\tfrac{3}{4}) + 7g(1)\right]$$

$$\Rightarrow \int_a^b f(x)\,dx = \frac{(b-a)}{90}\left[7g(0) + 32g(\tfrac{1}{4}) + 12g(\tfrac{1}{2}) + 32g(\tfrac{3}{4}) + 7g(1)\right],$$

where $g(y) = f(y(b-a)+a)$.

$\boxed{7.2.8}$ Since $\int_0^1 ae^x + b\cos(\pi x/2) = (e-1)a + \frac{2}{\pi}b$

and $f(0) = a+b$, $f(1) = ae$.

thus, only need to find A_0 and A_1, s.t. $A_0 f(0) + A_1 f(1) = (e-1)a + \frac{2}{\pi}b$.

Consider the system $\begin{cases} A_0 + eA_1 = e-1 \\ A_0 = \frac{2}{\pi} \end{cases}$

$$\Rightarrow \begin{cases} A_0 = \frac{2}{\pi} \\ A_1 = \dfrac{e-1-\frac{2}{\pi}}{e}. \end{cases}$$

$\boxed{7.2.12}$ Solve by method of undetermined coefficients

(5)

Consider: $\int_1^3 f(x)dx = A_0 f(0) + A_1 f(2) + A_2 f(4)$.

Use $1, x, x^2$ as tests functions.

we have

$$2 = \int_1^3 1 dx = A_0 + A_1 + A_2$$

$$4 = \int_1^3 x dx = 2A_1 + 4A_2$$

$$\frac{26}{3} = \int_1^3 x^2 dx = 4A_1 + 16A_2$$

$$\Rightarrow \begin{cases} A_0 + A_1 + A_2 = 2 \\ 2A_1 + 4A_2 = 4 \\ 4A_1 + 16A_2 = \frac{26}{3} \end{cases} \Rightarrow \begin{cases} A_0 = \frac{1}{12} \\ A_1 = \frac{11}{6} \\ A_2 = \frac{1}{12} \end{cases}$$

$$\Rightarrow \int_1^3 f(x)dx = \frac{1}{12} f(0) + \frac{11}{6} f(2) + \frac{1}{12} f(4).$$