MATH 170B Mid-Term I

1. (a) 2nd order Taylor series:

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$$f(x) \approx f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 \quad 2$$

where $f'(x) = 2\sin x \cos x$, $f''(x) = 2\cos^2 x - 2\sin^2 x$. 2

thus, $f(x) \approx 0 + 0 + \frac{2}{2}x^2 = x^2$ 2

6 (b). The error is $|f(\frac{\pi}{6}) - P_2(\frac{\pi}{6})|$

$$= |R_3(\tfrac{\pi}{6})| \quad , \text{ since } f^{(3)}(x) = -8\sin x \cos x = -4\sin(2x) \quad 2$$

$$= \left|\frac{f^{(3)}(\xi)}{3!}(\tfrac{\pi}{6})^3\right| \quad 2, \quad \xi \in [0, x]$$

$$= \left|\frac{4\sin(2\xi)}{6}(\tfrac{\pi}{6})^3\right| \leq \frac{2}{3}(\tfrac{\pi}{6})^3. \quad 2$$

(c). Actual error $= |f(\frac{\pi}{6}) - P_2(\frac{\pi}{6})|$

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$$= |\sin^2(\tfrac{\pi}{6}) - (\tfrac{\pi}{6})^2| \quad 2$$

$$= (\tfrac{\pi}{6})^2 - \tfrac{1}{4}. \quad 1$$

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2. Recall that the error of bisection method is bounded as following:

$$|r - c_n| \leq 2^{-(n+1)} (b_0 - a_0) \leq \varepsilon \quad 5$$

then, let's try to solve for n.

$$-(n+1)\log 2 \leq \log \varepsilon - \log(b_0 - a_0)$$

$$n+1 \geq \frac{\log(b_0 - a_0) - \log \varepsilon}{\log 2}$$

$$n \geq \frac{\log(b_0 - a_0) - \log \varepsilon}{\log 2} - 1 \quad 5$$

3. (a) Take $\lim_{n\to\infty}$ on both sides of

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$$x_{n+1} = 2x_n - a x_n^2$$

we have $\quad r = 2r - ar^2$

Case (i) $r = 0$

Case (ii) $r \neq 0 \Rightarrow ar = 1$

$$r = \frac{1}{a}.$$

(b) Only consider the ~~a~~ case with $r = \frac{1}{a}$ 3

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To find $\frac{1}{a}$, one can transform it as a root finding Problem $f(x) = 0$, where $f(x) = a - \frac{1}{x}$

thus, the Newton's iteration regarding to f is ~~2~~ 1

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}, \quad f'(x) = \frac{1}{x^2} \quad 1$$

$$\Rightarrow X_{n+1} = X_n - \frac{a - \frac{1}{x_n}}{\frac{1}{x_n^2}} = X_n - aX_n^2 + X_n$$

$$= 2X_n - aX_n^2. \quad 1$$

(c). To compute $\sqrt[3]{R}$, one can find

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the root $f(x) = 0$ with $f(x) = x^3 - R$ 2

thus the Newton's algorithm is

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}, \quad \text{with } f'(x) = 3x^2$$

$$\Rightarrow X_{n+1} = X_n - \frac{X_n^3 - R}{3X_n^2} = \frac{2}{3}X_n + \frac{R}{3X_n^2}. \quad 2$$

starting the loop with a closed $X_0 \simeq \sqrt[3]{R}$.

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4 (a) Apple the CMT

(4)

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(i) Check $F: \mathbb{C} \to \mathbb{C}$ where $\mathbb{C} = (-\infty, +\infty)$ 1

Clearly $F(x) \in [-\frac{1}{2}-\frac{1}{5}, \frac{1}{2}+\frac{1}{5}] \subset \mathbb{R}$ 2

(ii) Show F is a contractive mapping.

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Use MVT.

$|F(x)-F(y)| = |F'(\xi)|\,|x-y|, \quad \forall x,y \in \mathbb{R},\ \xi \in \mathbb{R}$

Since $F'(x) = -\frac{1}{2}\sin x + \frac{2}{5}\cos 2x$ 2

$\Rightarrow |F(x)-F(y)| \le |\frac{1}{2}+\frac{2}{5}|\,|x-y| = \frac{9}{10}|x-y|.$ 2

define $\lambda = \frac{9}{10} < 1$, 1

hence, F is contractive mapping.

By i) ii), F has a fixed-pt on $\mathbb{R}$.

(b). Consider $|F(x)-F(y)| = |G(x)-G(y) + \frac{1}{2}\cos(x) + \frac{1}{5}\sin(2x) - \frac{1}{2}\cos(y) - \frac{1}{5}\sin(2y)|$

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$\le |G(x)-G(y)| + \frac{9}{10}|x-y|$ 3, $\forall x,y \in \mathbb{R}$

In order to sufficiently guarantee there is a fixed-pt of F,

we can make a condition on G, that is

"G is a contractive mapping with contractive constant $\lambda < \frac{1}{10}$"

that is $|G(x)-G(y)| \le \lambda\,|x-y|, \ \forall x,y \in \mathbb{R}$.

2 with $\lambda < \frac{1}{10}$ 2