1. (a) 2nd order Taylor series:

\[ f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 \]

where \( f'(x) = 2\sin x \cos x \), \( f''(x) = 2\cos^2 x - 2\sin x \).

thus, \( f(x) \approx 0 + 0 + \frac{2}{2} x^2 = x^2 \)

(b) The error is \( |R_2(\frac{\pi}{6})|\)

\[ = |R_3(\frac{\pi}{6})|, \text{ since } f^{(3)}(x) = -8\sin x \cos x \]

\[ = \left| \frac{f^{(3)}(\xi)}{3!}(\frac{\pi}{6})^3 \right|, \quad \exists \xi \in [0, x] \]

\[ = \left| \frac{4\sin(2\xi)}{6}(\frac{\pi}{6})^3 \right| \leq \frac{2}{3}(\frac{\pi}{6})^3 \]

(c) Actual error = \( |f(\frac{\pi}{6}) - P_2(\frac{\pi}{6})|\)

\[ = |\sin(\frac{\pi}{6}) - (\frac{\pi}{6})^2|/2 \]

\[ = (\frac{\pi}{6})^3 - \frac{1}{4} \]
2. Recall that the error of the bisection method is bounded as following:

\[ |r - C(n)| \leq 2^{-c(n+1)} (b_0 - a_0) \leq 3 \]

Then, let's try to solve for \( n \):

\[-(n+1) \log 2 \leq \log 3 - \log (b_0 - a_0)\]

\[ n + 1 \geq \frac{\log (b_0 - a_0) - \log 3}{\log 2} \]

\[ n \geq \frac{\log (b_0 - a_0) - \log 3}{\log 2} - 1 \]

3. (a) Take \( \lim_{h \to 0} \) on both sides of:

\[ x_{n+1} = 2x_n - ax_n^2 \]

We have:

\[ r = 2r - ar^2 \]

Case (i) \( r = 0 \)

Case (ii) \( r \to 0 \) \( \Rightarrow \) \( ar = 1 \)

\[ r = \frac{1}{a} \]
(b) Only consider the case with \( r = \frac{1}{a} \).

To find \( a \), one can transform it as a root finding problem \( f(x) = 0 \), where \( f(x) = x^2 - a \).

Thus, the Newton's iteration regarding to \( f \) is

\[
X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}, \quad f'(x) = \frac{1}{x^2}
\]

\[
\Rightarrow X_{n+1} = X_n - \frac{a - \frac{1}{x_n}}{\frac{1}{x_n}} = X_n - aX_n^2 + X_n
\]

\[
= 2X_n - aX_n^2
\]

(c) To compute \( 3^{\sqrt{R}} \), one can find the root \( f(x) = 0 \) with \( f(x) = x^3 - R \).

Thus, the Newton's algorithm is

\[
X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}, \quad \text{with} \quad f'(x) = 3x^2
\]

\[
\Rightarrow X_{n+1} = X_n - \frac{x_n^3 - R}{3x_n^2} = \frac{2}{3}X_n + \frac{R}{3x_n^2}
\]

starting the loop with a closed \( X_0 \) for \( 3^{\sqrt{R}} \).
4. (a) Apply the CMT

(i) Check $F: \mathbb{R} \rightarrow \mathbb{R}$ where $\mathbb{R} = (-\infty, +\infty)$

Clearly $F(x) \in [\frac{-1}{2} - \frac{1}{3}, \frac{1}{2} + \frac{1}{3}] \subset \mathbb{R}$

(ii) Show $F$ is a contraction mapping.

(v) Use MVT.

$$|F(x) - F(y)| = |F'(\xi)| |x - y|, \quad \forall x, y \in \mathbb{R}, \quad \xi \in \mathbb{R}$$

Since $F'(x) = -\frac{1}{2} \sin x + \frac{2}{3} \cos 2x$

$$\Rightarrow |F(x) - F(y)| \leq \frac{1}{2} + \frac{2}{3} |x - y| = \frac{9}{20} |x - y|$$

Define $\lambda = \frac{9}{20} < 1$.

hence, $F$ is a contraction mapping.

By (i), (ii), $F$ has a fixed-point on $\mathbb{R}$.

(b) Consider

$$|F(x) - F(y)| = |G(x) - G(y)| + \frac{1}{2} \cos x + \frac{1}{3} \sin 2x$$

$$-\frac{1}{4} \cos (y) - \frac{1}{3} \sin (2y)|$$

$$\leq |G(x) - G(y)| + \frac{9}{20} |x - y|, \quad \forall x, y \in \mathbb{R}$$

In order to sufficiently guarantee there is a fixed-point of $F$,

we can make a condition on $G$, that is

"$G$ is a contraction mapping with contraction constant $\lambda < \frac{1}{10}$ that is $|G(x) - G(y)| \leq \lambda |x - y|, \quad \forall x, y \in \mathbb{R}$ with $\lambda < \frac{1}{10}$"