Mid-Term I

1. (a) $f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f^{(3)}(x)}{3!}h^3 + \cdots + \frac{f^{(n)}(x)}{n!}h^n + \cdots$

+10

Or $= \sum_{i=0}^{\infty} \frac{f^{(i)}(x)}{i!} h^i$

(b) $f(x+h) = \underbrace{f(x) + f'(x)h + \cdots + \frac{f^{(n-1)}(x)}{(n-1)!}h^{n-1}}_{n \text{ terms}} + \frac{f^{(n)}(\xi)}{n!}h^n$

+10

where $\xi \in (x, x+h)$.

2. (a) Let $f(x) = x^4 - x^3 - 10$.

+6

Then Since $f(2) = -2 < 0$, $f(3) = 44 > 0$

Thus in $[2,3]$, there exists a solution.

+7

(b).

	a_n	$f(a_n)$	c	$f(c)$	b	$f(b)$
step 0:	2	-2	2.5	13.4375	3	44
step 1:	2	-2	2.25	4.2383	2.5	13.4375
step 2:	2	-2	$\boxed{2.125}$		2.25	4.2383

$\Rightarrow r_2 = 2.125$.

+7

(c). Solve Ineq: $\frac{b-a}{2^{n+1}} = \frac{1}{2^{n+1}} \le 10^{-10}$

$\Rightarrow 2^{n+1} > 10^{10}$

$\Rightarrow (n+1)\log 2 > 10$

$\Rightarrow n > 10/\log 2 - 1 \approx 32.22$

So totally $\boxed{33}$ steps are required.

3. (a) Truncate ~~the~~ Taylor series as following ②

+5 Let r be the root, i.e $f(r)=0$

$$0=f(r)=f[x+(r-x)]=f(x)+f'(x)(r-x)+\cdots$$

$$\Rightarrow \quad r-x=-\frac{f(x)}{f'(x)}, \quad \text{def } r-x=h,$$

then $r=x+h$, where $h=-\dfrac{f(x)}{f'(x)}$.

(b). $f(x)=x^2-x-1$, $f'(x)=2x-1$, $x_0=0$

+5 step 1: $x_1=x_0-\dfrac{f(x_0)}{f'(x_0)}=0-\dfrac{-1}{-1}=-1$

step 2: $x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}=-1-\dfrac{1}{-3}=-\dfrac{2}{3}$.

(c) $x=x^2-1$, where $g(x)=x^2-1$.

+5

(d) $x_1=x_0^2-1=-1$

+5 $x_2=x_1^2-1=0$

4. $A=\begin{bmatrix}2 & 0\\ -1 & 1\end{bmatrix}$

i) $\|A\|_\infty=\max\limits_{i}\{\sum\limits_{i=1}^{2}|a_{ij}|, j=1,2$

+4 $=2$

ii) Since $A^{-1}=\begin{bmatrix}\frac{1}{2} & 0\\ \frac{1}{2} & 1\end{bmatrix} \Rightarrow \|A^{-1}\|_\infty=\frac{3}{2}$

+8 $\Rightarrow \text{Cond}(A)=\|A\|\cdot\|A^{-1}\|=3$.

(iii) Eigenvalues.

~~8~~ +6

Solve $\det(A-\lambda I)=0$

One has $(\lambda-2)(\lambda-1)=0$

$\Rightarrow \lambda_1=2, \quad \lambda_2=1.$

(iv) Spectral radius

~~8~~ +2

$\rho(A)=\max_i |\lambda_i| = 2.$

5. Jacobi: $X_{k+1}=D^{-1}(b-(L+U)X_k)$, where $D=\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}$, $b=\begin{bmatrix}5\\4\end{bmatrix}$

+10

$L=\begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix}$, $U=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}$

$\Rightarrow X_1 = D^{-1}(b-(L+U)X_0)$

$= D^{-1}b = \begin{bmatrix}5/3\\ -2\end{bmatrix}$

$\Rightarrow X_2 = D^{-1}(b-(L+U)X_1)$

$= \begin{bmatrix}1\\ -17/6\end{bmatrix}$

GS: ~~$X_{k+1}=$~~ $(D+L)X_{k+1}=b-UX_k$

+16

$\Rightarrow X_1 = \begin{bmatrix}5/3\\ -17/6\end{bmatrix}$

$\Rightarrow X_2 = \begin{bmatrix}13/18\\ -85/36\end{bmatrix}.$