1. (a) (3)

$$\ell_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x(x-1)}{(-1)(-2)} = \frac{x(x-1)}{2}.$$

$$\ell_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{1\cdot(-1)} = -(x+1)(x-1)$$

$$\ell_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)(x-0)}{2\cdot 1} = \frac{x(x+1)}{2}.$$

(4)

(b) $P_2(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + f(x_2)\ell_2(x)$

$$= \frac{x(x-1)}{2} - (x+1)(x-1) + \frac{3x(x+1)}{2}$$

$$= x^2+x+1$$

(6) (c).

x	$f(x)$		
-1	1	0	1
0	1	2	
1	3		

(3)

$$P_2(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1)$$

$$= 1 + 0(x+1) + 1(x+1)x$$

$$= x^2+x+1$$ (3)

2. i) $p(x_0) = a + b(x_0 - x_0) + f[x_0, x_0, x_1](x_0 - x_0)^2$

2

$= a$

ii) By Thm 5 on Page 347

4

$$f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}$$

$$= \frac{\frac{f(x_1) - f(x_0)}{x_1 - x_0} - \frac{f'(x_0)}{1!}}{x_1 - x_0} = \frac{c-a}{(x_1-x_0)^2} - \frac{b}{x_1 - x_0}$$

thus, $P(x_1) = a + b(x_1 - x_0) + \left(\frac{c-a}{(x_1-x_0)^2} - \frac{b}{x_1-x_0}\right)(x_1 - x_0)^2$

$= c.$

2

iii) $p'(x) = b + 2f[x_0, x_0, x_1](x - x_0)$

$\Rightarrow p'(x_0) = b + 2 \cdot \left(\frac{c-a}{(x_1-x_0)^2} - \frac{b}{x_1-x_0}\right)(x_0 - x_0)$

$= b.$

3. For quadratic spline, $f(x)$ is cont 1.

and $f'(x)$ is cont.

3

i) $S_0(1)=1$ $S_1(2)=\frac{3}{2}$ 2

$S_1(1)=1$ $S_2(2)=\frac{3}{2}$

ii) $S_0'(X)=1$, $S_1'(X)=2-X$, $S_2'(X)=0$

$\Rightarrow S_0'(1)=1$ $S_1'(2)=0$ 2

$S_1'(1)=1$ $S_2'(2)=0$.

Hence, f is a quadratic spline function.

4. The knots are 1, 2, 3, 4 and 5 w.s.r.p to $B_2^3(X)$ with $k=3$.

By Lemma 2 on Page 368.

the support of $B_1^3(X)$ is $(1, 5)$ includes $X_1=4.1$

$B_2^3(X)$ is $(2\ \ 6)$ includes $X_2=4.5$

$B_3^3(X)$ is $(3\ \ 7)$ includes $X_3=4.6$ 3

$B_4^3(X)$ is $(4\ \ 8)$ includes $X_4=6.1$

$B_5^3(X)$ is $(5\ \ 9)$. includes $X_5=7.6$.

Thus, all $B_j^3(X_j)\neq 0$, $j=1,\dots,5$.

3

By Schoenberg-Whitney Thm, A_{ij} is non-singular, thus we can use these nodes.

4

5. (a) $f(x) - P(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$, $n=2$ 2

4

$$= \frac{1}{3!} f^{(3)}(\xi_x)(x-x_0)(x-x_1)(x-x_2)$$

$$= \frac{1}{6} f^{(3)}(\xi_x)(x+1)x(x-1)$$ 2

(b). $f^{(3)}(x) = -\cos x$

4 $|f(x) - P(x)| = \left| \frac{1}{6} \cos x \, (x+1)x(x-1) \right|$

$$\le \left| \frac{1}{6} (x+1)(x)(x-1) \right|$$ 2

~~Since~~ let $g(x) = x(x-1)(x+1) = x^3 - x$

consider $g'(x) = 0 \Rightarrow x = \pm \frac{\sqrt{3}}{3}$

try $g(x)$ on $\pm\frac{\sqrt{3}}{3}, -1, 1$

the max of g on $[-1,1]$ is $\frac{2\sqrt{3}}{9}$

$\Rightarrow |f(x) - P(x)| \le \frac{\sqrt{3}}{27}$. 2

(c). By Thm 5 on page 318

4

Since $|x| \le 1$

$$|f(x) - P(x)| \le \frac{1}{2^2(2+1)!} \cdot \max_{|t|\le 1} |f^{(3)}(t)|$$ 2

$$\le \frac{1}{2^2 3!} = \frac{1}{24}.$$ 2

5

6. Since $B_j^0(x) = \begin{cases} 1 & \text{if } t_i \le x < t_{i+1} \\ 0 & \text{otherwise.} \end{cases}$

i) Obviously, $B_j^0(x) = D_j^0(x+s)$. +2

ii) Assume $k=n$, $B_j^n(x) = D_j^n(x+s)$. +1

Consider $k=n+1$, by eq (1) on page 367

$$D_j^{n+1}(x+s) = \frac{x+s-u_j}{u_{j+n+1}-u_j} D_j^n(x+s) + \frac{u_{j+n+2}-x-s}{u_{j+n+2}-u_{j+1}} D_{j+1}^n(x+s).$$

+1

Notice that, $D_j^n(x+s) = B_j^n(x)$, $D_{j+1}^n(x+s) = B_{j+1}^n(x)$

$u_{j+n+1}-u_j = t_{j+n+1}-t_j$, $u_{j+n+2}-u_{j+1} = t_{j+n+2}-t_{j+1}$,

$x+s-u_j = x-(u_j-s) = x-t_j$, $u_{j+n+2}-s-x = t_{j+n+2}-x$.

Plug these into the above eq, +1

$$D_j^{n+1}(x+s) = \frac{x-t_j}{t_{j+n+1}-t_j} B_j^n(x) + \frac{t_{j+n+2}-x}{t_{j+n+2}-t_{j+1}} B_{j+1}^n(x)$$

$$= B_j^{n+1}(x) \quad \text{by eq (1).}$$

By Induction, $B_j^k(x) = D_j^k(x+s)$ +1 for all integer $k \ge 0$.