MID-TERM II

1\. (a) Use Lagrange Interpolation $P_2(x) = \sum_{i=0}^{2} f(x_i)\, \ell_{2i}(x)$

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where $P_2(x) = 0 \cdot \cancel{\ell_0(x)}^{\,0} + 2 \cdot \frac{x(x-2)}{(1-0)(1-2)} + 10\, \frac{x(x-1)}{(2-0)(2-1)}$

$= -2x(x-2) + 5x(x-1)$

$= 3x^2 - x$

(b).

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x	$f[x]$		
0	0		
		2	
1	2		3
		8	
2	10		

$\Rightarrow P_2(x) = 0 + 2x + 3x(x-1)$

$= 3x^2 - x.$

(c). $f(\frac{1}{2}) \approx P_2(\frac{1}{2}) = \frac{1}{4}$

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(d). $\text{Error} = |f(x) - P_2(x)| = \left| \frac{x(x-1)(x-2)}{3!} f^{(3)}(c) \right|, \quad c \in [0, 2]$

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(e) Since $f'''(x) = 6$

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$\Rightarrow \quad \text{Error} = |x(x-1)(x-2)|$

if $x \in [0, 1]$, $|x(x-1)| \le \frac{1}{4}$, $|x-2| \le 2 \Rightarrow \text{Error} \le \frac{1}{2}$

if $x \in [1, 2]$, $|x| \le 2$, $|(x-1)(x-2)| \le \frac{1}{4} \Rightarrow \text{Error} \le \frac{1}{2}$

Hence, upper bound is $\boxed{\frac{1}{2}}$.

2. (a) $N_1(f,h) = \dfrac{f(x+h) - f(x)}{h}$

By Taylor series, $f(x+h) = f(x) + hf'(x) + O(h^2)$

$$\Rightarrow \quad f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

N_1 is to approximate $f'(x)$ 6 with accuracy $O(h)$ 6

(b) $N_2(f,h) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

By Taylor Series,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + O(h^4)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + O(h^4)$$

Sum them up, we have

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4)$$

$$\Rightarrow \quad f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

N_2 is to approximate $f''(x)$ with accuracy $O(h^2)$

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3. (a). By Lagrange Interpolation

$$P_1(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}$$ 7

then $$\int_a^b P_1(x) = \frac{f(a)}{a-b}\int_a^b (x-b) + \frac{f(b)}{b-a}\int_a^b (x-a)$$

$$= \frac{f(a)}{a-b}\,\frac{(a-b)(b-a)}{2} + \frac{f(b)}{b-a}\,\frac{(b-a)^2}{2}$$

$$= \frac{b-a}{2}\left(f(a) + f(b)\right).$$ 8

(b). The error term is

$$e(x) = \frac{(x-a)(x-b)}{2} f''(c(x))$$ 3

Notice that ~~this~~ value of this random c depends on variable x.

thus $$\int_a^b e(x)dx = \int_a^b \frac{(x-a)(x-b)}{2} f''(c(x))dx$$

Since $(x-a)(x-b)$ does not change sign, by MVT, 3

$$\int_a^b e(x)dx = \frac{f''(\xi)}{2}\int_a^b (x-a)(x-b)dx, \quad \xi \in [a,b].$$

Let $x-a=u$, then

$$\int_a^b (x-a)(x-b)dx = \int_0^h u(u-h)\,du, \quad h = b-a$$

$$= -\frac{h^3}{6}$$

$$\Rightarrow \left|\int_a^b e(x)\,dx\right| = \frac{h^3}{12}|f''(\xi)|, \quad \xi \in [a,b].$$ 4

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(a) Assume $F(x+h)=0$

8 By Taylor series, $F(x+h) = F(x) + J(x)\cdot\vec{h} + O(h^2) = 0$

then $\vec{h} \approx -J^{-1}(x)F(x)$

then the root $x+h = x - J^{-1}(x)F(x)$

(b) Assume $\lim_{i\to\infty} x_i = x$,

7 then i) Linear convergence: $\lim_{i\to\infty} \frac{|x_{i+1}-x|}{|x_i - x|} = \mu,\ 0<\mu<1.$

ii) Superlinear convergence: $\lim_{i\to\infty} \frac{|x_{i+1}-x|}{|x_i - x|} = 0$

iii) quadratic convergence: $\lim_{i\to\infty} \frac{|x_{i+1}-x|}{|x_i - x|^2} = \mu,\ \mu>0$

(c) 10 $J(x) = \begin{bmatrix} 1 & -2x_2 \\ -2x_1 & 1 \end{bmatrix}$

step 1: $J(x_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $F(x_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, ~~let~~

Solve $J(x_0)h = -F(x_0)$ for h, $\Rightarrow h = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = x_0 + h = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

step 2: $J(x_1) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $F(x_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Solve $J(x_1)h = -F(x_1)$ for h, $\Rightarrow h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow x_2 = x_1 + h = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.