DIMENSION THEORY

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Contents

1. Introduction

The following discussion is taken from the Stacks project and reformatted to fit the purposes of our seminar.

We want to understand what it means to assign a dimension to a commutative ring R. Equivalently, we want to understand the dimension of the topological space $\mathrm{Spec}(R)$. In algebraic geometry, the correct way to do this is via the Krull dimension.

Definition 1.1. Let R be a commutative ring. The Krull dimension of R is

$$\sup\{n : \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n\}$$

where the $\mathfrak{p}_i$ are prime ideals and the inclusions are strict. Equivalently, the Krull dimension of $\mathrm{Spec}(R)$ is

$$\sup\{n : X_0 \subset X_1 \subset \cdots \subset X_n\}$$

where the X_i are irreducible closed subsets of $\mathrm{Spec}(R)$ and inclusions are strict.

The first reasonable finiteness assumption is that we should assume our rings are Noetherian. This assumption at least gives us some hope of having a reasonable theory.

Remark 1.2. There are Noetherian rings with infinite Krull dimension. These rings are pathological, and don't really come up in geometry. We will ignore these for the most part. The reason these exist is, even though each chain has to be finite, there can be infinitely many chains, each increasing in length.

The definition of Krull dimension is nice, but it is difficult to use it to compute. Our first computational tool will be the Hilbert polynomial. For finite type graded algebras over a field, these results were due essentially to Hilbert. We will work in a more general situation.

2. Noetherian Graded Rings and Numerical Polynomials

Our graded rings will be $\mathbb{N}$-graded, and $0 \in \mathbb{N}$.

Lemma 2.1. *Let S be a graded ring. Let S_+ be the irrelevant ideal. Then a set of elements $f_i \in S_+$ generate S as an S_0-algebra if and only if they generate S_+ as an ideal of S.*

Proof. Suppose the elements $f_i \in S_+$ generate S as an S_0-algebra. Then every element of S_+ is a polynomial in the f_i without constant term. Hence, the f_i generate S_+ as an ideal. Conversely, suppose S_+ is generated by the f_i as an ideal of S. It suffices to prove that every homogeneous element $f \in S$ is a polynomial in the f_i with coefficients in S_0. Say f has degree d. We do induction on d. The case $d = 0$ is obvious, so suppose $d > 0$. Then $f \in S_+$, so by assumption we can write

$$f = \sum g_i f_i$$

for some $g_i \in S$. Replace g_i by its homogeneous component of degree $d - \deg(f_i)$. Then by induction, we're done. □

Lemma 2.2. *A graded ring S is Noetherian if and only if S_0 is Noetherian and S_+ is finitely generated as an ideal of S.*

Proof. If S is Noetherian, then certainly S_0 is as it is a quotient, $S_0 \cong S/S_+$, of a Noetherian ring. Moreover, S_+ is finitely generated because S is Noetherian. Conversely, if S_0 is Noetherian and S_+ is finitely generated as an ideal of S, take a finite set of generators $f_1, \ldots, f_n$. By the previous lemma, these generators generate S as an S_0-algebra. That is, we have a surjection

$$S_0[X_1, \ldots, X_n] \twoheadrightarrow S$$

where the X_i go to f_i. By the Hilbert basis theorem $S_0[X_1, \ldots, X_n]$ is Noetherian, so we must have that S is Noetherian, as we can view it as the quotient of a Noetherian ring. □

Now we have some nice criterion for when our graded rings are Noetherian. We want to now introduce numerical polynomials. Later, we'll show that the Hilbert polynomial (whatever that is) is a numerical polynomial.

Definition 2.3. Let A be an abelian group. A function $f : n \to f(n) \in A$ defined for all sufficiently large integers n is a numerical polynomial if there exists an $r \geq 0$ and elements $a_0, \ldots, a_r \in A$ such that

$$f(n) = \sum_{i=0}^{r} \binom{n}{i} a_i$$

Lemma 2.4. *Let $f : n \to f(n) \in A$ be defined for all sufficiently large integers n. If $n \mapsto f(n) - f(n-1)$ is a numerical polynomial, then so is f.*

Proof. Do it yourself! □

3. Digression on the Group $K_0^{'}(R)$

Definition 3.1. Let R be a commutative ring. The group $K_0^{'}(R)$ is the group determined by the following properties:

(1) For every finitely generated R-module M, there is an element $[M] \in K_0^{'}(R)$.

(2) For every short exact sequence of R-modules
$$0 \to M' \to M \to M'' \to 0$$
we have the relation $[M] = [M'] + [M'']$
(3) The generators are the symbols $[M]$, and every relation is a $\mathbb{Z}$-linear combination of the relations coming from the short exact sequence relations.

For the purposes of this seminar, we need to get a handle on this group for some simple examples.

Lemma 3.2. *Let R be a principal ideal domain. Then $K_0^{'}(R) \cong \mathbb{Z}$.*

Proof. By the structure theorem for finite modules over a principal ideal domain, we have that for any R-module M,
$$M \cong R^r \times R/(d_1) \times \cdots \times R/(d_n)$$
Consider the short exact sequence
$$0 \to (d_i) \to R \to R/(d_i) \to 0$$
Then in the group $K_0^{'}(R)$, we have $[R] = [(d_i)] + [R/(d_i)]$. But (d_i) is a free R-module, so as a module it is isomorphic to R. Thus we have that
$$[R/(d_i)] = 0$$
Therefore, there is a natural map $K_0^{'}(R) \to \mathbb{Z}$ sending a module to its rank. It's clear that this is an isomorphism. □

Example 3.3. $K_0^{'}(\mathbb{Z}) = K_0^{'}(k) = K_0^{'}(k[x]) = \mathbb{Z}$ where k is a field.

4. Our First Example of a Numerical Polynomial

We want to show the following theorem:

Theorem 4.1. *Let S be a Noetherian graded ring. Let M be a finite graded S-module. If S_+ is generated by elements in degree 1, then the function*
$$\mathbb{Z} \to K_0'(S_0)$$
given by $n \to [M_n]$ is a numerical polynomial.

For this theorem to even make sense, we have to show that the $[M_n]$ are finite S_0 modules.

Lemma 4.2. *If M is a finite S-module, and if S is finitely generated over S_0 as an algebra, then each M_n is a finite S_0-module.*

Proof. Take homogeneous sets of generators, m_i for M, and f_i for S. We can assume that the $f_i \in S_+$. Then each M_n is generated by elements of the form $\prod f_i^{e_i} m_i$ whose degree is n. □

Now we know the theorem statement makes sense, so let's prove it.

Proof. We proceed by induction on the minimal number of generators of S_1. If this number is 0, then $M_n = 0$ for all $n >> 0$ and we're done. Now let $x \in S_1$ be an element of a minmial set of generators of S_1. First, suppose that x is nilpotent. That is, there exists some r such that $x^r M = 0$. Now we proceed by induction on

r. If $r = 1$, then M is an S/xS-module. So the result holds by the first induction hypothesis. If $r > 1$ then there exists as short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

with the property that there exist r' and r'', both less than r, with $x^{r'}M' = 0$ and $x^{r''}M'' = 0$. Thus, by induction, the result is true for M' and M''. In the group $K_0'(S_0)$, we have the relation

$$[M_d] = [M_d'] + [M_d'']$$

so we get the result for M.

Now suppose x is not nilpotent. Then let $M' \subset M$ be the largest submodule of M on which x is nilpotent. Then we have a short exact sequence

$$0 \to M' \to M \to M/M' \to 0$$

We get the relation $[M_n] = [M'] + [M/M']$. By the argument in the nilpotent case, it thus suffices to prove the result for $[M/M']$. That is, we can assume x acts injectively. Then, for each n, we have the short exact sequence

$$0 \to M_d \to M_{d+1} \to (M/xM)_{d+1} \to 0$$

Thus, we have $[(M/xM)_{d+1}] = [M_{d+1}] - [M_d]$. The left hand side gives us a numerical polynomial by induction. Thus, the right hand side is a numerical polynomial. Now just apply Lemma 2.4, and we're done. □

References

[1] The Stacks Project Authors. *Stacks Project*.
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