(1) How many positive integers \( \leq 1000 \) are neither perfect squares nor perfect cubes? [Recall that a perfect square is an integer of the form \( n^2 \) where \( n \) is an integer, and a perfect cube is an integer of the form \( n^3 \) where \( n \) is an integer.]

(2) Let \( \lambda \) be an integer partition. Write \( \lambda \subseteq m \times n \) if \( \ell(\lambda) \leq m \) and \( \lambda_1 \leq n \), i.e., the Young diagram of \( \lambda \) fits inside of a \( m \times n \) rectangle. For \( 0 < k < n \), define a polynomial \( P_{n,k}(x) \) by

\[
P_{n,k}(x) = \sum_{\lambda \subseteq k \times (n-k)} x^{\ell(\lambda)}.
\]

In other words, the coefficient of \( x^i \) is the number of partitions of \( i \) whose Young diagram fits into the \( k \times (n-k) \) rectangle. By convention, \( P_{n,n}(x) = P_{n,0}(x) = 1 \).

As an example, \( P_{4,2}(x) = 1 + x + 2x^2 + x^3 + x^4 \) (the 1 corresponds to the fact that there is a single partition of size 0).

(a) Show that \( P_{n,k}(x) = P_{n,n-k}(x) \).

(b) If \( 0 < k < n \), show that

\[
P_{n,k}(x) = x^k P_{n-1,k}(x) + P_{n-1,k-1}(x).
\]

(c) Using (b), show that \( P_{n,k}(1) = \binom{n}{k} \) for all \( 0 \leq k \leq n \).

[If you cannot solve (b), you can still use it to solve this problem for credit.]

(d) Find a direct explanation (not using (b)) for why \( P_{n,k}(1) = \binom{n}{k} \). In other words, show that the number of Young diagrams that fit inside the \( k \times (n-k) \) rectangle is \( \binom{n}{k} \).

(3) How many ways are there to list the letters of the word WISCONSIN so that no two consecutive letters are the same?

(4) Fix a positive integer \( n \). Show that

\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.
\]

(5) Fix a positive integer \( n \). Show that

\[
\sum_{k=0}^{n} \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = -\frac{1}{n+1}.
\]
Hints:
2(b): Think about adding/removing columns from Young diagrams
2(c): You can either do a double induction, or do induction on $n + k$.
2(d): Given a Young diagram $Y(\lambda) \subseteq k \times (n - k)$, we can remove it, and the top boundary of the resulting shape is a path from the bottom left corner of the rectangle to the top right corner using the steps “up” and “right”. Show these are counted by $\binom{n}{k}$.
5: Antiderivative