Math 222, Fall 2016
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Notes for Nov. 29 lecture

Problem 1. Show that $2 < e < 3$.

By plugging in $x = 1$ into the equation $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, we have

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

Each term $\frac{1}{k!}$ is positive, and the first two terms are 1, so we get $e > 2$.

To get the upper bound, we use the inequality $n! \geq 2^{n-1}$ for all $n \geq 1$.[1] Taking reciprocals, this says $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ (the inequality is actually strict for $n \geq 3$). So we can also get a bound

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots.$$

Everything after the first 1 is actually just the geometric series $\sum_{k=0}^{\infty} \frac{1}{2^k}$. From an earlier example, we know that

$$\frac{1}{1-x} = T_\infty \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{for } |x| < 1.$$

So plug in $x = 1/2$, and see that our geometric series is $\frac{1}{1-\frac{1}{2}} = 2$. In conclusion, $e < 3$.

Problem 2. Show that $T_\infty \arctan x = \arctan x$ for $0 \leq x \leq 1$.

To do that, we will show that

$$\lim_{n\to\infty} |(R_n \arctan)(x)| = 0 \qquad \text{for } |x| \leq 1.$$

The idea is that $\arctan x$ is the antiderivative of $\frac{1}{1+x^2}$, so we should start from information about that. Start with[2]

$$\frac{1}{1-u} = T_n \frac{1}{1-u} + \frac{u^{n+1}}{1-u}.$$

Now substitute $u = -t^2$:

$$\frac{1}{1+t^2} = T_{2n} \frac{1}{1+t^2} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}.$$

Now integrate from 0 to x:

$$\arctan x = T_{2n+1} \arctan x + \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt.$$

So the integral is $R_{2n+1} \arctan x$. But also, the Taylor series of arctan has no even powers of x in it, so it's also equal to $R_{2n+2} \arctan x$. In symbols:

$$R_{2n+1} \arctan x = R_{2n+2} \arctan x = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt.$$

[1] To see why it is true, note that $n!$ is a product of $n-1$ numbers: $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2$ and each one is ≥ 2.

[2] This uses that $T_n \frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n = \frac{1-u^{n+1}}{1-u}$.

So the sequence $R_n \arctan x$ is repeating every 2 terms, so its limit can be computed just from the odd terms, or just the even ones. (Introducing redundancy into a sequence won't change its limit.)

You can evaluate this integral using long division and the different methods from Chapter 1, but that's more work than is needed. We just want to bound the remainder.

We will use this fact: $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ if $a \le b$.

So we want to bound the absolute value of the integrand. Since $1+t^2 \ge 1$ for any t, we get

$$\left|\frac{(-1)^{n+1}t^{2n+2}}{1+t^2}\right| \le |(-1)^{n+1}t^{2n+2}| = |t|^{2n+2}.$$

Let's put it together:

$$\begin{aligned}
|R_{2n+1}\arctan x| &= \left|\int_0^x \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}dt\right| \\
&\le \int_0^x \left|\frac{(-1)^{n+1}t^{2n+2}}{1+t^2}\right| dt \\
&\le \int_0^x |t|^{2n+2}dt \\
&= \frac{x^{2n+3}}{2n+3}.
\end{aligned}$$

Finally,

$$\lim_{n\to\infty}|R_n \arctan x| = \lim_{n\to\infty}|R_{2n+1}\arctan x| \le \lim_{n\to\infty}\frac{x^{2n+3}}{2n+3} \le \lim_{n\to\infty}\frac{1}{2n+3} = 0$$

(the first equality is what we said before about the sequence just repeating every 2 terms and the second inequality is because $x \le 1$). By the sandwich theorem, this also says that $\lim_{n\to\infty}|R_n \arctan x| = 0$, so we're done.

As a consequence, we get the formula[3]

$$\frac{\pi}{4} = \arctan(1) = T_\infty \arctan(1) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1},$$

which can be rewritten as

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right).$$

Bonus: Show that $T_\infty \arcsin(x) = \arcsin(x)$ for $x = \frac{1}{2}$. This gives another infinite sum formula (though more complicated) for π since $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$.

[3] $\tan(\pi/4) = 1$ can be obtained by looking at a right isosceles triangle