Math 222, Fall 2016
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Review packet solution outline
(last updated 12/17/16)

1.

$$\begin{aligned} \frac{dy}{dx} &= x^{3/2}(4-y^2)^{3/2} \\ \int \frac{dy}{(4-y^2)^{3/2}} &= \int x^{3/2} dx \end{aligned}$$

For left side do trig sub: $y = 2\sin\theta$, $\sqrt{4-y^2} = 2\cos\theta$:

$$\begin{aligned} \int \frac{2\cos\theta d\theta}{8\cos^3\theta} &= \frac{2}{5}x^{5/2} + C \\ \frac{1}{4}\tan\theta &= \frac{2}{5}x^{5/2} + C \\ \frac{y}{4\sqrt{4-y^2}} &= \frac{2}{5}x^{5/2} + C \end{aligned}$$

Initial condition is $y(0) = 0$, so $C = 0$.

$$\begin{aligned} y &= \frac{8}{5}x^{5/2}\sqrt{4-y^2} && \text{(clear denominators)} \\ y^2 &= \frac{64}{25}x^5(4-y^2) && \text{(square both sides)} \\ y^2 + \frac{64}{25}x^5y^2 &= \frac{256}{25}x^5 \\ y^2 &= \frac{256}{25}\frac{x^5}{1+\frac{64}{25}x^5} \\ y &= \frac{16x^{5/2}}{\sqrt{25+64x^5}} \end{aligned}$$

2.

$$\begin{aligned} T_\infty \frac{1}{1-t} &= \sum_{n=0}^{\infty} t^n \\ T_\infty \frac{1}{1+x^4} &= \sum_{n=0}^{\infty} (-1)^n x^{4n} \\ T_\infty \frac{x}{1+x^4} &= \sum_{n=0}^{\infty} (-1)^n x^{4n+1} \end{aligned}$$

First, figure out where it converges with ratio test. The limit is $|x^4|$, so converges for $|x| < 1$ and diverges for $|x| > 1$. Test endpoints: both diverge.

So candidates for $T_\infty f(x) = f(x)$ are $|x| < 1$, so restrict attention here.

$$\begin{aligned} R_n \frac{1}{1-t} &= \frac{t^{n+1}}{1-t} \\ R_{4n} \frac{1}{1+x^4} &= \frac{(-1)^n x^{4n+4}}{1+x^4} \\ R_{4n+1} \frac{x}{1+x^4} &= \frac{(-1)^n x^{4n+5}}{1+x^4} \end{aligned}$$

Now take limit:

$$\lim_{n\to\infty} \left| R_{4n+1} \frac{x}{1+x^4} \right| = \frac{1}{|1+x^4|} \lim_{n\to\infty} |x|^{4n+5} = 0.$$

So $T_\infty f(x) = f(x)$ for $|x| < 1$.

3. (a)

$$\begin{aligned} \int \frac{x^2}{x^2-x-2} dx &= \int \left(1 + \frac{x+2}{x^2-x-2}\right) dx && \text{(long division)} \\ &= \int \left(1 + \frac{4/3}{x-2} - \frac{1/3}{x+1}\right) dx && \text{(partial fractions)} \\ &= x + \frac{4}{3} \ln|x-2| - \frac{1}{3} \ln|x+1| + C \end{aligned}$$

(b)

$$\begin{aligned} \int \frac{dx}{x(x-1)^2} &= \int \left(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right) dx && \text{(partial fractions)} \\ &= \ln|x| - \ln|x-1| - \frac{1}{x-1} + C && (u\text{-sub with } u = x-1) \end{aligned}$$

4.

$$\sum_{n=1}^{\infty} \frac{(1+e^n)e^n}{e^{3np}} = \sum_{n=1}^{\infty} \left(\frac{1}{(e^{3p-1})^n} + \frac{1}{(e^{3p-2})^n} \right)$$

Sum of two geometric series. First one converges if $e^{3p-1} > 1$, i.e., $3p - 1 > 0$, i.e., $p > 1/3$ and goes to $+\infty$ otherwise. Similarly, second one converges if $p > 2/3$ and goes to $+\infty$ otherwise. If at least one goes to ∞ it diverges ($\infty + \infty = \infty$), so need both to converge. This only happens when $p > 2/3$.

5. (a)

$$\frac{dy}{dx} + (\cot x)y = \sin x$$

First-order equation: $a(x) = \cot x = \frac{\cos x}{\sin x}$,

$$A(x) = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| \qquad \text{do } u\text{-sub with } u = \sin x$$

So $m(x) = e^{\ln \sin x} = |\sin x|$ is a multiplier. It's annoying to work with absolute values in our integrals, so let's make sure that $\sin x$ is also a multiplier (on the exam you can skip this step). Multiply both sides of our diffeq by $\sin x$:

$$(\sin x)\frac{dy}{dx} + (\cos x)y = \sin^2 x$$

Left side is the derivative of $(\sin x)y$, so $((\sin x)y)' = \sin^2 x$. Now integrate and divide by $\sin x$:

$$\begin{aligned} y &= \frac{1}{\sin x}\int \sin^2 x \; dx \\ &= \frac{1}{2\sin x}\int (1 - \cos 2x)dx && \text{(double-angle formula)} \\ &= \frac{1}{2\sin x}\left(x - \frac{\sin(2x)}{2} + C\right) \end{aligned}$$

(b) First-order: $a(x) = 1$, $A(x) = x$, $m(x) = e^x$, $k(x) = \frac{\sin^3 x \cos^2 x}{e^x}$.

$$\begin{aligned} y &= e^{-x}\int \sin^3 x \cos^2 x dx \\ &= e^{-x}\int \sin x(1 - \cos^2 x)\cos^2 x dx \\ &= -e^{-x}\int (u^2 - u^4)du && \text{(u-sub with $u = \cos x$)} \\ &= -e^{-x}\left(\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + C\right) \end{aligned}$$

6. Use Taylor polynomial to approximate $f(x) = \ln(1 + x)$ at $x = 1/2$. Need remainder to be $< 0.02 = \frac{1}{50}$. Trial and error shows $n = 1, 2$ doesn't work. For $n = 3$, $f^{(4)}(\xi) = \frac{-6}{(1+\xi)^4}$. When we apply Lagrange, we will have $0 \le \xi \le 1/2$, so $|f^{(4)}(\xi)| \le 6$.

$$|(R_3 f)(\frac{1}{2})| = \frac{|f^{(4)}(\xi)|(1/2)^4}{4!} \le \frac{6}{16 \cdot 24} = \frac{1}{64} < \frac{1}{50}.$$

So approximation is

$$(T_3 f)(\frac{1}{2}) = \frac{1}{2} - \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} = \frac{5}{12}.$$

7. (a)

$$\begin{aligned} T_5 e^{x^2} &= 1 + x^2 + \frac{x^4}{2} \\ T_5 \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \end{aligned}$$

We want coefficient of x^5 of $T_\infty(e^{x^2}\sin x)$. To get that, multiply the above two and get coefficient of x^5. But you don't have to multiply everything out. The only combinations that give x^5 are $1 \cdot \frac{x^5}{5!}$, $x^2 \cdot (-\frac{x^3}{3!})$, and $\frac{x^4}{2} \cdot x$. Sum those up and get the coefficient is $\frac{1}{5!} - \frac{1}{3!} + \frac{1}{2}$.

So $\frac{f^{(5)}(0)}{5!} = \frac{1}{5!} - \frac{1}{3!} + \frac{1}{2}$, or $f^{(5)}(0) = 1 - 20 + 60 = 41$.

(b)

$$\begin{aligned} T_\infty \frac{1}{1-t} &= \sum_{n=0}^{\infty} t^n \\ T_\infty \frac{1}{(1-t)^2} &= \sum_{n=0}^{\infty} n t^{n-1} && \text{(take derivative)} \\ T_\infty \frac{1}{(1+x^3)^2} &= \sum_{n=0}^{\infty} n(-1)^{n-1} x^{3n-3} && (\text{sub } t = -x^3) \end{aligned}$$

We get x^{27} when $n = 10$, so $\frac{f^{(27)}(0)}{27!} = -10$, or $f^{(27)}(0) = -10 \cdot 27!$.

8. Use ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(n+1)^x e^{n+1}}{(2n+2)!} \frac{(2n)!}{n^x e^n} \right| &= \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right)^x \frac{e}{(2n+2)(2n+1)} \right| \\ &= e \lim_{n\to\infty} \left| \left(1 + \frac{1}{n}\right)^x \frac{1}{(2n+2)(2n+1)} \right| \end{aligned}$$

$(1 + \frac{1}{n})^x \to 1$ when $n \to \infty$ and the second term $\frac{1}{(2n+2)(2n+1)} \to 0$ when $n \to \infty$, so the limit is always 0. So the series converges for all x.

9. Produce two vectors parallel to plane by taking difference of points:

$$\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Take cross product to get a normal vector (note there are many other options):

$$\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

Defining equation: $5x - y + 3z = 0$.

For second part, the normal vector $\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$ is parallel to the normal line (by definition) so (one possible) parametric equation is

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}.$$

10. Use integration by parts with $du = x^{-n}dx$ and $v = \sin x$ to get

$$\int \frac{\sin x}{x^n}dx = \frac{\sin x}{x^{n-1}(1-n)} - \frac{1}{1-n}\int \frac{\cos x}{x^{n-1}}dx$$

Do integration by parts again with $du = x^{1-n}$ and $v = \cos x$ to get

$$\int \frac{\cos x}{x^{n-1}}dx = \frac{\cos x}{x^{n-2}(2-n)} - \frac{1}{2-n}\int \frac{-\sin x}{x^{n-2}}dx$$

Put them together:

$$\int \frac{\sin x}{x^n}dx = \frac{\sin x}{x^{n-1}(1-n)} - \frac{\cos x}{x^{n-2}(1-n)(2-n)} - \frac{1}{(1-n)(2-n)}\int \frac{\sin x}{x^{n-2}}dx$$

Alternatively:

$$I_n = \frac{\sin x}{x^{n-1}(1-n)} - \frac{\cos x}{x^{n-2}(1-n)(2-n)} - \frac{1}{(1-n)(2-n)}I_{n-2}$$

(There is a typo in the answer in the review packet)

The improper integral $\int_0^\infty \frac{\sin x}{x^3}dx$ has two forms of improperness (at 0 and ∞), so split it up as $\int_0^1 \frac{\sin x}{x^3}dx + \int_1^\infty \frac{\sin x}{x^3}dx$ (the choice of 1 is arbitrary). Examine the first one (and replace it with $\lim_{a\to 0^+}\int_a^1 \frac{\sin x}{x^3}dx$).

Using our reduction formula:

$$\lim_{a\to 0^+}\int_a^1 \frac{\sin x}{x^3}dx = \lim_{a\to 0^+}\left(\left[\frac{\sin x}{-2x^2} - \frac{\cos x}{2x}\right]_a^1 - \frac{1}{2}\int_a^1 \frac{\sin x}{x}dx\right)$$

The function $\frac{\sin x}{x}$ is actually continuous at 0, so its integral from 0 to 1 is some finite number. So we have to see if

$$\lim_{a\to 0^+}\left(\frac{\sin a}{-2a^2} - \frac{\cos a}{2a}\right)$$

exists or not.

$$\begin{aligned}
\lim_{a\to 0^+}\left(\frac{\sin a}{-2a^2} - \frac{\cos a}{2a}\right) &= \lim_{a\to 0^+} -\frac{\sin a + a\cos a}{2a^2} \\
&= \lim_{a\to 0^+} -\frac{\cos a + \cos a - a\sin a}{4a} \qquad \text{(l'Hôpital's rule)} \\
&= -\frac{2}{0} = -\infty
\end{aligned}$$

So the limit does not exist, and our improper integral diverges.

11.

$$T_\infty e^{-t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!}$$
$$T_\infty t^k e^{-t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{n+k}}{n!}$$
$$T_\infty f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+k+1}}{n!(n+k+1)}$$

Now let's deal with remainders:

$$R_n f(x) = \int_0^x R_{n-1}(t^k e^{-t})dt = \int_0^x t^k R_{n-1-k} e^{-t} dt$$

By Lagrange, $R_{n-k-1}e^{-t} = \dfrac{(-1)^{n-k}e^{-\xi}t^{n-k}}{(n-k)!}$ for some ξ between 0 and t. Put it together:

$$R_n f(x) = \int_0^x \frac{(-1)^{n-k}e^{-\xi}t^n}{(n-k)!} dt.$$

Hard to work with since ξ depends on t (so is not a constant). We should split up cases of $x \geq 0$ and $x \leq 0$. First consider $x \geq 0$. Then $0 \leq \xi \leq t \leq x$, so $|e^{-\xi}| \leq 1$, and

$$|R_n f(x)| \leq \int_0^x \left| \frac{(-1)^{n-k}e^{-\xi}t^n}{(n-k)!} \right| dt \leq \int_0^x \frac{t^n}{(n-k)!} dt = \frac{x^{n+1}}{(n+1)(n-k)!}.$$

The limit as $n \to \infty$ is 0 since factorial beats exponentials. So we showed that $T_\infty f(x) = f(x)$ for $x \geq 0$.

Now consider $x \leq 0$. In that case we should rewrite

$$R_n f(x) = \int_0^x \frac{(-1)^{n-k}e^{-\xi}t^n}{(n-k)!} dt = -\int_x^0 \frac{(-1)^{n-k}e^{-\xi}t^n}{(n-k)!} dt$$

Now $0 \geq \xi \geq t \geq x$, so $|e^{\xi}| \geq |e^x|$, or $|e^{-\xi}| \leq |e^{-x}|$. So:

$$|R_n f(x)| \leq \int_x^0 \left| \frac{(-1)^{n-k}e^{-\xi}t^n}{(n-k)!} \right| dt \leq e^{-x} \int_x^0 \frac{|t^n|}{(n-k)!} dt = e^{-x} \frac{|x|^{n+1}}{(n-k)!}.$$

Again, the limit as $n \to \infty$ is 0 since factorial beats exponentials. So we also showed that $T_\infty f(x) = f(x)$ for $x \leq 0$.

12. (a) Use alternating series. First, we check the terms go to 0. Replace the sequence by the function $\dfrac{\ln x}{\sqrt{x}}$ and use l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

Now, we check the terms are decreasing. Not easy to do directly, so need to check where the derivative of $\frac{\ln x}{\sqrt{x}}$ is negative. The derivative is $\frac{2 - \ln x}{2x^{3/2}}$. The numerator is negative when $\ln x \geq 2$ (which happens for $x \geq 9$ since $e^2 < 9$) and the denominator is positive for $x \geq 9$, so the whole thing is negative for $x \geq 9$, so $\frac{\ln x}{\sqrt{x}}$ is decreasing once $x \geq 9$, and hence the same is true for the sequence.

Alternating series test then says the series $\sum_{n=9}^{\infty} \frac{(-1)^n \ln(n)}{\sqrt{n}}$ converges, and the same is true for our original one by the tail theorem.

(b) Use limit comparison test (the terms are always positive) against $\sum_{n=1}^{\infty} \frac{n^4}{n^5}$:

$$\lim_{n\to\infty} \frac{n^4 + 6n - 3 + \cos n}{n^5 + 3n^3 - 2n + 1} \cdot \frac{n^5}{n^4} = \frac{1 + \frac{6}{n^3} - \frac{3}{n^4} + \frac{\cos n}{n^4}}{1 + \frac{3}{n^2} - \frac{2}{n^4} + \frac{1}{n^5}} = 1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test, so same is true for our series.

13. If $\vec{v}$ and $\vec{w}$ are orthogonal, then their dot product is 0, i.e., $2x + 2y - 2 = 0$. Solve for y: $y = 1 - x$. Also, $\|\vec{w}\| = 3$, and $\|\vec{v}\| = \sqrt{x^2 + y^2 + 4}$, so need $x^2 + y^2 + 4 = 9$, or $x^2 + y^2 = 5$. Plug in $y = 1 - x$: $x^2 + 1 - 2x + x^2 = 5$, or $2x^2 - 2x - 4 = 0$. Factor that: $2(x - 2)(x + 1) = 0$, so our solutions are $x = 2$ and $x = -1$. In each case, $y = -1$ and $y = 2$.

To summarize: the two vectors $\vec{v}$ that satisfy the conditions are $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.