Math 742, Spring 2016
Homework 11
Due: April 22

1. Exercises

(1) A field k is **perfect** if either k has characteristic 0, or k has characteristic p, and the equation $x^p - a = 0$ has a solution for all $a \in k$.
  (a) Show that finite fields and algebraically closed fields are perfect.
  (b) Show that k is perfect if and only if every algebraic extension of k is separable.

(2) (a) Let E be a field of characteristic different from 2. Let $F = E(\sqrt{D})$ be a quadratic extension (so $D \in E$ but $\sqrt{D} \notin E$). Show that $\mathrm{N}_E^F(a + b\sqrt{D}) = a^2 - Db^2$ and that $\mathrm{Tr}_E^F(a + b\sqrt{D}) = 2a$.
  (b) Apply Hilbert's theorem 90 to the extension $\mathbf{Q}(i)/\mathbf{Q}$ to show that all solutions to $a^2 + b^2 = 1$ with $a, b \in \mathbf{Q}$ are of the form

$$a = \frac{s^2 - t^2}{s^2 + t^2}, \qquad b = \frac{2st}{s^2 + t^2}$$

  for some $s, t \in \mathbf{Q}$.
  (c) Let D be an integer such that $\sqrt{D}$ is not an integer. Generalize (b) to determine the solutions to $a^2 - Db^2 = 1$ for $a, b \in \mathbf{Q}$.

(3) Read Example 7 from Lang, §VI.2. Use it to show that the Galois groups of these two polynomials (over $\mathbf{Q}$) is the full symmetric group S_4:

$$x^4 + 2x^2 + x + 3, \qquad x^4 + 3x^3 - 3x - 2.$$

(4) Let K/k be a finite separable extension of degree n. Write $K = k(\theta)$ where $\theta \in K$ and let $\theta_1, \ldots, \theta_n$ be the conjugates of θ in $\overline{k}$. Define an equivalence relation $\theta_i \sim \theta_j$ if $k(\theta_i) = k(\theta_j)$ as subfields of $\overline{k}$.
  (a) Show that the equivalence classes all have the same size.
  (b) Now assume n is prime. Conclude that either all of the subfields are the same or are all different. In particular, show that if K contains two of the θ_i, then in fact K/k is a cyclic extension.

2. Further reading

I've chosen not to put any complicated Galois group calculations on the homework. If you'd like to do some, there are plenty of resources for that. For example, you might read Section 14.8 of Dummit, Foote, *Abstract Algbera* (third edition) and look at its exercises.