Math 742, Spring 2016
Homework 3
Due: February 12

1. Exercises

(1) Let I, J be ideals of a ring R and let M be an R-module.
   (a) Show that $R/I \otimes_R M \cong M/IM$.
   (b) Show that $R/I \otimes_R R/J \cong R/(I+J)$.
   (c) In particular, show that $\mathbf{Z}/n \otimes_{\mathbf{Z}} \mathbf{Z}/m = 0$ if n, m are relatively prime integers.

(2) Let R be a local ring with maximal ideal $\mathfrak{m}$. Let M, N be finitely generated R-modules.
   (a) Show that if $M \otimes_R (R/\mathfrak{m}) = 0$, then $M = 0$.
   (b) Show that if $M \otimes_R N = 0$, then either $M = 0$ or $N = 0$.

(3) Show that the tensor product of two free R-modules is again free. More specifically, given sets Λ, Λ', construct an isomorphism[1]

$$R^{\oplus \Lambda} \otimes_R R^{\oplus \Lambda'} \xrightarrow{\cong} R^{\oplus(\Lambda \times \Lambda')}.$$

Conclude that the tensor product of two projective R-modules is again projective.

(4) Given any module M, its **dual** is the module $M^\vee = \mathrm{Hom}_R(M, R)$. Define a map $\sigma_M\colon M \to (M^\vee)^\vee$ by $\sigma_M(m)(f) = f(m)$ (where $m \in M$ and $f \in M^\vee$). Show that σ_M is an isomorphism if M is a finitely generated projective module.

(5) Let P, P' be finitely generated projective modules and let M, M' be arbitrary modules. Construct an isomorphism

$$\mathrm{Hom}_R(P, M) \otimes_R \mathrm{Hom}_R(P', M') \xrightarrow{\cong} \mathrm{Hom}_R(P \otimes_R P', M \otimes_R M').$$

Deduce the following isomorphisms:

$$P^\vee \otimes_R (P')^\vee \xrightarrow{\cong} (P \otimes_R P')^\vee,$$
$$P^\vee \otimes_R M' \xrightarrow{\cong} \mathrm{Hom}_R(P, M').$$

(6) Let R be a ring and let M be an R-module. Show that if M is flat, then M is torsion-free[2]. Show that the converse is true if R is a principal ideal domain.

Conclude that if M is finitely generated and R is a PID, then M is flat if and only if M is free.

(7) Let R be an integral domain and let $\alpha\colon R^n \to R^n$ be an endomorphism. Show that if α is nilpotent, i.e., $\alpha^r = 0$ for some $r > 0$, then in fact, $\alpha^n = 0$. Give an example to show that this statement can fail if R is not required to be an integral domain.

[1] It's instructive to think about how to prove this only using universal mapping properties.
[2] In general, torsion-free means that if $m \in M$ is nonzero and $rm = 0$ for $r \in R$, then r is a zerodivisor.

2. Suggested exercises (don't submit)

From Altman–Kleiman:

- Chapter 8: 12, 19, 24, 26
- Chapter 9: 4, 8, 10, 12, 13, 25
- Chapter 10: 8, 9, 14, 16

3. Further reading

The Tor functor (and more generally, derived functors) are not treated in the text. This is a useful tool for studying tensor products when the modules aren't flat, but takes more time to develop than I think we'll have. Some discussion can be found in the last page of §XVI.3 of Lang, but it can also be found in any textbook which treats homological algebra.