Math 742, Spring 2016
Homework 5
Due: February 26

1. Exercises

(1) Let k be a field and let $\mathfrak{m} = (x-1)$ be a maximal ideal in $k[x]$. Let $Q = \mathfrak{m} \cap k[x^2-1]$ where $k[x^2-1]$ is the subring generated by x^2-1. Show that $1/(x+1) \in k[x]_{\mathfrak{m}}$ is not integral over $k[x^2-1]_Q$.

This shows that even if $S \subset R$ is integral, $R_{\mathfrak{m}}$ need not be integral over $S_{S\cap\mathfrak{m}}$.

(2) Let K be a field extension of the field k. A subset $S \subset K$ is **algebraically independent over** k if, given any nonzero polynomial $p(x_1, \dots, x_r)$ with coefficients in k and distinct $s_1, \dots, s_r \in S$, we have $p(s_1, \dots, s_r) \neq 0$. Order the set of algebraically independent subsets by inclusion. If S is a maximal algebraically independent subset, then S is a **transcendence basis**.

- (a) Use Zorn's lemma to show that transcendence bases exist.
- (b) If S is a finite algebraically independent subset and T is another algebraically independent subset with $|T| > |S|$, show that there exists $x \in T$ such that $S \cup \{x\}$ is also algebraically independent.
- (c) Now assume that K is finitely generated over k (i.e., K is the fraction field of a finitely generated k-algebra). Show that K has a finite transcendence basis over k. Conclude from (b) that all transcendence bases have the same size; this number d is the **transcendence degree** of K over k.

  In particular, if K is algebraic over a subfield generated by d elements, then these elements must be algebraically independent.

(3) Let k be a field and let $R = k[x_1, \dots, x_n]$. In class, we used the fact that $\dim R_{\mathfrak{m}} = n$ for all maximal ideals $\mathfrak{m}$ of R without proving it. The goal of this exercise is to show that $\dim R_{\mathfrak{m}}$ is independent of $\mathfrak{m}$ without using the full version of (15.1), so don't quote any results including and after (15.9) from the text for this problem (also don't use (15.1)).

- (a) Show that $\dim R_{\mathfrak{m}}$ is independent of $\mathfrak{m}$ when k is algebraically closed.
- (b) Show that $\dim R_{\mathfrak{m}}$ is independent of $\mathfrak{m}$ in general by considering the integral extension $R \subset \overline{k}[x_1, \dots, x_n]$ where $\overline{k}$ is an algebraic closure of k.

(4) Give an example of a non-noetherian ring such that every ascending chain of prime ideals $P_1 \subseteq P_2 \subseteq \cdots$ stabilizes, i.e., $P_j = P_{j+1} = \cdots$ for some j. This shows that the obvious variation of Theorem 16.10 is not true.

2. Suggested exercises (don't submit)

From Altman–Kleiman:

- Chapter 14: 4, 13, 14, 17
- Chapter 15: 11, 17, 19