Math 742, Spring 2016
Homework 8
Due: April 1

1. Exercises

Notation: Unless otherwise stated, i is a complex square root of -1.
$\mathbf{Q}$ is the field of rational numbers.

(1) Show that $\sqrt{2}+\sqrt{3}$ is algebraic over $\mathbf{Q}$ of degree 4. Find its minimal polynomial.

(2) Let E, F be extensions of a field k which are both in a common field K. Show that

$$[EF : k] \leq [E : k][F : k].$$

Assuming both $[E : k]$ and $[F : k]$ are finite and relatively prime, show that the above inequality is actually an equality.

(3) Let $f(x) \in k[x]$ be a polynomial of degree n and let K be its splitting field. Show that $[K : k]$ divides $n!$.

(4) Let α be a real number such that $\alpha^4 = 5$.
 (a) Show that $\mathbf{Q}(i\alpha^2)/\mathbf{Q}$ is normal.
 (b) Show that $\mathbf{Q}(\alpha + i\alpha)/\mathbf{Q}(i\alpha^2)$ is normal.
 (c) However, show that $\mathbf{Q}(\alpha + i\alpha)/\mathbf{Q}$ is not normal.

(5) Describe the splitting field and calculate the degree of the extension over $\mathbf{Q}$ of the following polynomials:
 (a) $x^4 - 2$
 (b) $x^4 + x^2 + 1$
 (c) $x^6 - 4$

2. Suggested reading

I assume that you've seen basics of polynomials in an undergraduate algebra course and we won't say much about them explicitly. In particular, you may want to review Lang, §IV.3 on irreducibility criteria.