Math 742, Spring 2016
Takehome exam 1
Due: March 28 (MONDAY) by the end of class (2:10PM)

- Do not discuss this exam with anyone else.
- You may use your notes and textbook and use as much time as you like up until the due date. You are free to use any results in the book that we covered and any previous exercises.
- No late exams will be accepted. If you cannot make it to class, you can email me your solutions.
- This counts for 4 homework assignments and all problems will be graded. Each problem will be scored out of 10 points, so the maximum score is 60 points.

1. Exercises

(1) Let S be a subring of R.
   (a) Give an example where R is noetherian and S is not noetherian.
   (b) Assume there is an S-module map $\phi\colon R \to S$ which is the identity when restricted to S. Show that S is noetherian if R is noetherian.

(2) Let R be a ring and let $\phi\colon R^n \to R^n$ be a linear map where n is finite.
   (a) Show that $\det \phi \in \operatorname{ann}(\operatorname{coker} \phi)$.
   (b) Give an example where $\det \phi$ is a nonzerodivisor but $\operatorname{ann}(\operatorname{coker} \phi)$ is not equal to the ideal generated by $\det \phi$.

(3) A ring is **reduced** if its nilradical is 0.
   (a) Show that a ring R is reduced if and only if $R_{\mathfrak{m}}$ is reduced for all maximal ideals $\mathfrak{m}$.
   (b) Give an example of a field k and a reduced k-algebra R such that $R \otimes_k \overline{k}$ is not reduced where $\overline{k}$ is an algebraic closure of k.

(4) Let k be a field. In each case, verify that R is an integral domain and describe, as explicitly as possible, its normalization.
   (a) $R = k[x, y]/(x^4 - y^3)$
   (b) $R = k[x, y, z]/(x^2 - yz)$

(5) Let R be a noetherian ring and let M be a finitely generated R-module. Recall in HW3 #4, we defined the dual $M^\vee = \operatorname{Hom}_R(M, R)$ and the map $\sigma_M\colon M \to (M^\vee)^\vee$ by $\sigma_M(m)(f) = f(m)$ (where $m \in M$ and $f \in M^\vee$).
   (a) Show that σ_M is injective if and only if M is isomorphic to a submodule of $R^{\oplus n}$ for some finite n.
   (b) Show that σ_M is bijective if and only if there exists a homomorphism $\phi\colon R^{\oplus n} \to R^{\oplus m}$ for some finite n, m such that $M \cong \ker \phi$.

(6) Let $S \subset R$ be an integral extension.
   (a) Show that $\dim S = \dim R$.
   (b) Assume that R is a finitely generated S-module. Let $\mathfrak{p} \subset S$ be a prime ideal. Show that there are only finitely many prime ideals $\mathfrak{q} \subset R$ such that $\mathfrak{q} \cap S = \mathfrak{p}$.