Math 742, Spring 2016
Takehome exam 2
Due: May 13, 12pm (slide under my office door, 321 VV)

- Do not discuss this exam with anyone else.
- You may use your notes and textbook and use as much time as you like up until the due date. You are free to use any results in the book that we covered and any previous exercises.
- No late exams will be accepted. If necessary, you can email me your solutions.
- This counts for 4 homework assignments and all problems will be graded.

1. Exercises

(1) Let k be a field. Define $k^{\mathrm{fra}}(\!(t)\!)$ to be the set of formal expressions $\eta = \sum_{n \ge n_0} a_n t^{n/N}$ where $a_n \in k$, n_0 is an integer, and N is a positive integer, which depends on η.[1] You can add and multiply these expressions in the same way you add and multiply power series and it turns out that $k^{\mathrm{fra}}(\!(t)\!)$ is a field (you don't need to prove this).
   (a) Let $k(\!(t)\!)$ be the field of Laurent series, i.e., the subfield consisting of expressions such that the denominator N must be 1. Show that $k^{\mathrm{fra}}(\!(t)\!)$ is an algebraic extension of $k(\!(t)\!)$.
   (b) A harder fact is that if k is algebraically closed of characteristic 0, then $k^{\mathrm{fra}}(\!(t)\!)$ is also algebraically closed (you don't need to prove this). But this is false in positive characteristic: Suppose $k = \overline{\mathbf{F}}_p$. Show that $x^p - x - \frac{1}{t}$ has no solution in $k^{\mathrm{fra}}(\!(t)\!)$.

(2) (a) Let E be a field of characteristic different from 2. Suppose F is a Galois extension of E such that the Galois group is isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$. Show that there exist $x, y \in E$ such that $F = E(\sqrt{x}, \sqrt{y})$ such that none of x, y, xy are squares in E.
   (b) Conversely, pick $x, y \in E$ such that none of x, y, xy are squares in E. Show that $E(\sqrt{x}, \sqrt{y})/E$ is Galois with Galois group $\mathbf{Z}/2 \times \mathbf{Z}/2$.

(3) (a) Let G be a finite group. Show that there exists a Galois extension F/E in every characteristic whose Galois group is isomorphic to G.
   (b) Let G be a finite abelian group. Show that there exists $n > 0$ and a subfield $K \subset \mathbf{Q}(\zeta_n)$ (ζ_n is a primitive nth root of unity) such that $K/\mathbf{Q}$ is Galois with Galois group G.

(4) Let $f(x) = x^4 + ax^2 + b \in \mathbf{Q}[x]$ be an irreducible polynomial with splitting field K.
   (a) Show that $\mathrm{Gal}(K/F)$ is a subgroup of D_4 (the dihedral group of size 8).
   (b) Show that the Galois group is D_4 for $x^4 - 4x^2 - 1$.

(5) Let A be an alternating $n \times n$ matrix with entries in a field K such that $\mathrm{image}(A) = \ker(A)$ (both thought of as subspaces of K^n). In particular, n is even. Define a symmetric bilinear form $\langle , \rangle$ on K^n by

$$\langle (v_1, \dots, v_n), (w_1, \dots, w_n) \rangle = \sum_{i=1}^{n} v_i w_i.$$

   (a) Given a subspace $W \subseteq K^n$, define the subspace

$$W^\perp = \{x \in K^n \mid \langle x, w \rangle = 0 \text{ for all } w \in W\}.$$

[1] In other words, this is the set of power series where the exponents are allowed to be arbitrary rational numbers, provided that the exponents that appear are bounded from below and have bounded denominator.

Show that $\dim W + \dim W^{\perp} = n$.

(b) Let V be a subspace of K^n of dimension $n/2$. Define a bilinear form $(,)$ on V by

$$(v, w) = \langle v, Aw \rangle.$$

Show that $(,)$ is an alternating form.

(c) Show that if $\dim V = n/2$ and $V \cap \ker A = V \cap (\ker A)^{\perp} = 0$, then $(,)$ is symplectic (i.e., nondegenerate) on V.

(d) Show that such a V exists and conclude that n is divisible by 4.