Math 746, Spring 2016
Homework 2
Due: February 26

1. Exercises

(1) Eisenbud 18.12(a,b)

(2) Cohen–Macaulayness of graded rings can be used in combinatorics to prove positivity statements. Here's one simple positivity statement that can be proven without much machinery.

Let $R = \bigoplus_{n\geq 0} R_n$ be a noetherian graded algebra which is generated by R_1 over the field $k = R_0$. Assume that R has depth d with respect to the homogeneous ideal $R_+ = \bigoplus_{n>0} R_n$.

   (a) Show that if k is infinite, then there is a regular sequence of length d consisting of elements in R_1.

   (b) The Hilbert series of R can be written as
$$\sum_{n\geq 0}(\dim_k R_n)t^n = \frac{h_0 + h_1 t + \cdots + h_r t^r}{(1-t)^{\dim R}}.$$
If R is Cohen–Macaulay, show that $h_i \geq 0$ for $i = 0, \ldots, r$. (Reduce to the case where k is infinite and use (a)).

(3) Let k be a field and let n be a positive integer. Let $R = k[x_{ij} \mid 1 \leq i, j \leq n]$ be the polynomial ring in n^2 variables, which we think of as the polynomial functions on the set of $n \times n$ matrices $X = (x_{ij})$. Let $\mathcal{N}$ be the set of nilpotent matrices and let I be the ideal of all polynomial functions which vanish on $\mathcal{N}$.

   (a) Show that $\mathcal{N}$ is an irreducible algebraic set. (Depending on your background, you might not have enough to do this – so feel free to skip this part and use it for the remaining parts.)

   (b) Find n equations $f_1, \ldots, f_n$ with $\deg(f_i) = i$ so that a matrix is nilpotent if and only if each f_i vanishes on it. In particular, $\sqrt{(f_1, \ldots, f_n)} = I$. For example, when $n = 2$, $f_1 = \mathrm{trace}(X) = x_{1,1} + x_{2,2}$ and $f_2 = \det(X) = x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$.

   (c) Show that the Jacobian of $f_1, \ldots, f_n$ evaluated on a generic nilpotent matrix (i.e., having one Jordan block) has full rank n.

   (d) Conclude that $(f_1, \ldots, f_n) = I$ and hence R/I is Cohen–Macaulay.

   (e) Show that R/I satisfies (R_1)[1] and hence is normal. (Again, depending on background, you might not have enough to do this, so feel free to skip.)

2. Further reading

- Eisenbud 18.14 says that the ring of invariants R^G of a Cohen–Macaulay ring R is also Cohen–Macaulay in characteristic 0. For a proof, see Corollary 6.4.6 of Bruns, Herzog, *Cohen–Macaulay Rings*.
- Hochster's ICM talk on Cohen–Macaulay rings may also be of interest: `http://www.mathunion.org/ICM/ICM1978.1/Main/icm1978.1.0291.0298.ocr.pdf`.

[1] Geometrically: the singular locus of $\mathcal{N}$ has codimension ≥ 2 in $\mathcal{N}$