Math 746, Spring 2016
Homework 4
Due: April 11 (Monday)

1. Exercises

(1) Let m, n be positive integers and pick $1 \le r \le \min(m, n)$. Let R be a commutative ring. Set $S = R[x_{ij} \mid 1 \le i \le m,\ 1 \le j \le n]$. Let I_r be the ideal generated by the $r \times r$ minors of the generic $m \times n$ matrix $X = (x_{ij})$.

  (a) If R is a field, call a monomial ordering **anti-diagonal** if the initial monomial of a minor is the product of the entries along the anti-diagonal of the submatrix. For example, the anti-diagonal monomial of $\det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is $x_{12}x_{21}$. Show that anti-diagonal monomial orders exist.

  (b) Again, assuming R is a field, it is true that the $r \times r$ minors form a Gröbner basis for any anti-diagonal monomial ordering. Prove this in the case $r = m = 2$.[1] Conclude that $\mathrm{in}(I_r)$ is radical.

  (c) Use the fact in (b) to show that I_r is a prime ideal when R is a field.

  (d) Use the fact in (b) to show that S/I_r is a free R-module (first do the case $R = \mathbf{Z}$), and hence flat over R.

(2) Let R be a noetherian ring and let M be an R-module. Let $I, J \subset R$ be ideals.

  (a) Show that there is a natural exact sequence

$$0 \to \mathrm{H}^0_{I+J}(M) \to \mathrm{H}^0_I(M) \oplus \mathrm{H}^0_J(M) \to \mathrm{H}^0_{I\cap J}(M),$$

  and that the last map is surjective if M is an injective module.

  (b) Use injective resolutions to construct the Mayer–Vietoris sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{H}^0_{I+J}(M) & \longrightarrow & \mathrm{H}^0_I(M) \oplus \mathrm{H}^0_J(M) & \longrightarrow & \mathrm{H}^0_{I\cap J}(M) & \\ & & \swarrow & & & & \\ & \mathrm{H}^1_{I+J}(M) & \longrightarrow & \mathrm{H}^1_I(M) \oplus \mathrm{H}^0_J(M) & \longrightarrow & \mathrm{H}^1_{I\cap J}(M) & \longrightarrow \cdots \end{array}$$

(3) Let k be a field and $S = k[x_1, \dots, x_n]$. Let $I \subset S$ be an ideal. The **arithmetic rank** of I is the smallest r such that there exist $f_1, \dots, f_r$ with $\sqrt{(f_1, \dots, f_r)} = \sqrt{I}$.

  (a) If $\mathrm{H}^i_I(S) \neq 0$, show that the arithmetic rank of I is at least i.

  (b) Set $n = 4$ and $I = (x_1, x_2) \cap (x_3, x_4) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$. Use the Mayer–Vietoris sequence to show that $\mathrm{H}^3_I(S) \neq 0$.

  (c) Continue (b). Show that $I = \sqrt{(x_1x_3, x_2x_4, x_1x_4 + x_2x_3)}$ and conclude that I has arithmetic rank 3.[2]

[1] If you're feeling ambitious, do the general case, though I hope this special case already illustrates some interesting points.

[2] Give an argument for the radical equality that avoids the use of software.

2. Further reading

- The determinantal ring S/I_r is Cohen–Macaulay when R is Cohen–Macaulay; this essentially reduces to the case of a field, and in that case, one can prove this by showing that $S/\mathrm{in}(I_r)$ is Cohen–Macaulay. There are special techniques for showing that quotients by squarefree monomial ideals are Cohen–Macaulay. For example, see Theorem 5.53 of

  Ezra Miller, Bernd Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Math. **227**, Springer-Verlag, 2005.

  See Theorem 16.43 for a reference specific to this example.
- The arithmetic rank of any ideal in $k[x_1, \dots, x_n]$ with k a field is always at most n. For a proof and more general results, see:

  David Eisenbud, E. Graham Evans, Every algebraic set in n-space is the intersection of n hypersurfaces, *Invent. Math.* **19** (1973), 107–112.