MATH 490 HOMEWORK 1
DUE: FEBRUARY 2

You're strongly encouraged to write your homework in LaTeX. I won't require it for the first assignment, but I likely will for the rest of the assignments.

(1) Let M be an abelian group. Show that there is a unique way to give M the structure of a $\mathbf{Z}$-module ($\mathbf{Z}$ is the ring of integers).

(2) Let $\mathbf{M}$ be a complex of finite-dimensional vector spaces:

$$M_n \to M_{n-1} \to \cdots \to M_1 \to M_0$$

Show that

$$\sum_{i=0}^{n} (-1)^i \dim M_i = \sum_{i=0}^{n} (-1)^i \dim \mathrm{H}_i(\mathbf{M}).$$

(3) (a) Let G be the following oriented graph (I've labeled the vertices with coordinates).

$$\begin{tikzcd}
(0,1) \arrow[r, "\alpha_1"] \arrow[d, "\alpha_3"] & (1,1) \arrow[r, "\alpha_2"] \arrow[d, "\alpha_4"] & (2,1) \arrow[d, "\alpha_5"] \\
(0,0) \arrow[r, "\alpha_6"] & (1,0) \arrow[r, "\alpha_7"] & (2,0)
\end{tikzcd}$$

Write down the matrix representing the boundary homomorphism for G and compute a basis for $\mathrm{H}_1(G)$.

(b) More generally, pick positive integers m, n and let $G_{m,n}$ be the graph whose vertices are points (i, j) with $0 \le i \le m$ and $0 \le j \le n$. (The previous part concerns $G_{2,1}$.) All of the edges are between vertices "next to each other", i.e., (i, j) is connected to both $(i + 1, j)$ and $(i, j + 1)$.
Pick whatever orientation you like for the edges (explain it) and then compute a basis for $\mathrm{H}_1(G_{m,n})$. Also, give a simple formula for the size of the basis.
Part of the problem is to figure out a good way to label the edges (the way I did it in part (a) isn't so good – they don't have to all be called α!).

Hints:

(1) You know that you have to define $1 \cdot m = m$. What would $2 \cdot m$ have to be? $-1 \cdot m$?
(2) Try the case $n = 1$ first and then $n = 2$. The review exercises will help here.
(3) This problem can be solved using the results we proved in class, and is more about coming up with a good way to develop your own notation.

    For the general formula: if you can't deduce it, try fixing m and varying n and computing examples and see if you can find a pattern.