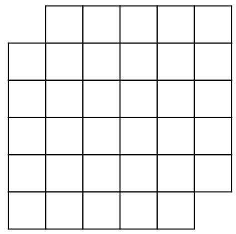
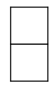
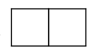
Math 184A, Fall 2018
Homework 1
Due: Friday, Oct. 5

(1) Take a 6×6 grid of squares of equal size and remove two diagonally opposite corner squares:



Prove that you cannot tile this with the shapes  and  without any overlaps.

(2) Prove that for all positive integers n,

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

(3) There are 17 marked points inside of an equilateral triangle of side length 1. Prove that there are 2 of them with distance at most 1/4 between them.

(4) Let $p(k)$ be a polynomial of degree d. The goal of this problem is to prove that $q(n) = \sum_{k=1}^{n} p(k)$ is a polynomial of degree $d+1$ which satisfies $q(0) = 0$.

Define $f^{(0)}(x) = 1$ and for each $d > 0$, define a degree d polynomial

$$f^{(d)}(x) = x(x-1)(x-2)\cdots(x-d+1).$$

(a) Show that the conclusion holds for each $f^{(d)}(x)$ with $d > 0$ by proving that

$$\sum_{k=1}^{n} f^{(d)}(k) = \frac{f^{(d+1)}(n+1)}{d+1}.$$

Verify the conclusion directly for $f^{(0)}(x)$.

(b) Show that if the conclusion holds for polynomials $p_1(x), \ldots, p_r(x)$, then it also holds for any linear combination $\alpha_1 p_1(x) + \cdots + \alpha_r p_r(x)$ (here α_i are scalars). You may assume that the $p_i(x)$ all have different degrees.

(c) Show, by induction on d, that any polynomial of degree d is a linear combination of $f^{(0)}(x), f^{(1)}(x), \ldots, f^{(d)}(x)$. For the induction step, note that if c is the leading coefficient of $p(x)$, then $p(x) - cf^{(d)}(x)$ is a polynomial of degree $\leq d-1$.