Math 184A, Fall 2018
Homework 5
Due: Friday, November 9 by 3:30PM in basement of AP&M

(1) Define a sequence by

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 3 \\ a_n &= 8a_{n-1} - 16a_{n-2} \qquad \text{for } n \geq 2. \end{aligned}$$

  (a) Express $A(x) = \sum_{n\geq 0} a_n x^n$ as a rational function in x.
  (b) Find a closed formula for a_n.

(2) Define a sequence of numbers $a_0, a_1, \dots$ by

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 2 \\ a_n &= -a_{n-1} + 2\sum_{i=0}^{n-2} a_i a_{n-2-i} \qquad \text{for } n \geq 2. \end{aligned}$$

Find a simple expression for $A(x) = \sum_{n\geq 0} a_n x^n$.

(3) If $\sum_{n\geq 0} a_n x^n = \dfrac{1+x+3x^3}{(1-2x)^4}$, find a formula for the a_n.

(4) Let $S(n,k)$ be the Stirling number of the second kind. For each $k \geq 1$, define the ordinary generating function

$$S_k(x) = \sum_{n\geq 0} S(n,k)x^n.$$

  (a) For $k \geq 2$, translate the identity from class
$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$$
  into an identity involving $S_k(x)$ and $S_{k-1}(x)$.
  (b) Use the identity you just found to show that for all $k \geq 1$, we have:
$$S_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

(5) If $p(n)$ is a polynomial of degree d, show that there is a polynomial $h_p(x)$ of degree $\leq d$ such that $\sum_{n\geq 0} p(n)x^n = \dfrac{h_p(x)}{(1-x)^{d+1}}$.

Hints:

5: If $p(n)$ is a polynomial of degree d, then $(\Delta p)(n) = p(n) - p(n-1)$ is a polynomial of degree $\leq d - 1$. What is the relationship between the generating function for $p(n)$ and $(\Delta p)(n)$?