Math 184A, Fall 2018
Homework 6
Due: Friday, November 16 by 3:30PM in basement of AP&M

(1) Let n be a positive integer and let a_n be the number of different ways to pay n dollars using only 1, 2, 5, 10, 20 dollar bills in which at most five 20 dollar bills are used. Express $A(x) = \sum_{n \geq 0} a_n x^n$ as a rational function.

(2) There are n aisles of shelves in a store. We want to separate them into consecutive nonempty groups for different categories of items. In addition, each category will be painted either red, blue, or green, and we will select some nonempty subset of the categories to be featured in the weekly advertisement. Let h_n be the number of ways to do this. Express $H(x) = \sum_{n \geq 0} h_n x^n$ as a rational function.

(3) Let a_n be the number of partitions of n so that each number which is used appears at least twice. For $n = 6$, the possible partitions are $(3,3), (2,2,2), (2,2,1,1), (1,1,1,1,1,1)$.

Let b_n be the number of partitions of n such that all parts have remainders in $\{0,2,3,4\}$ when divided by 6, i.e., the numbers which are not allowed are those of the form $1+6k$ and $5+6k$ where k is a non-negative integer. For $n = 6$, the possible partitions are $(6), (4,2), (3,3), (2,2,2)$.

Use generating functions to show that $a_n = b_n$ for all n.

(4) In class, we saw that the number of balanced strings of n pairs of parentheses is the **Catalan number** $C_n = \frac{1}{n+1}\binom{2n}{n}$ using generating functions. Now you will derive this formula avoiding generating functions.

Consider the set of paths from $(0,0)$ to (a,b) using the steps $(1,0)$ and $(0,1)$. In HW4, #2(d) (in the context of Young diagrams inside of a rectangle), we saw that there are $\binom{a+b}{b}$ of them. To be precise, a path is an ordered sequence of vectors $(v_1, \dots, v_{a+b})$ where each v_i is either $(1,0)$ or $(0,1)$ and $v_1 + \cdots + v_{a+b} = (a,b)$.

For example, for $(a,b) = (2,2)$, we draw below the two paths $((1,0),(0,1),(1,0),(0,1))$ and $((1,0),(0,1),(0,1),(1,0))$ by starting at $(0,0)$ and adding the vectors in order:

(The bottom left corner is $(0,0)$ and the top right corner is $(2,2)$.)

A path is **good** if it never goes strictly above the diagonal line $x = y$. In symbols, this means that the partial sums $v_1 + \cdots + v_i$ always have the property that the first coordinate is greater than or equal to the second coordinate for any $1 \leq i \leq 2n$. Any other path is **bad**.

For example, the path on the left in the example above is good while the path on the right is bad.

(a) Construct a bijection between the set of good paths from $(0,0)$ to (n,n) and the set of balanced strings of n pairs of parentheses.

(b) Given a bad path $(v_1, \dots, v_{2n})$ from $(0,0)$ to (n,n), let r be the smallest index such that $v_1 + \cdots + v_r$ is above the line $x = y$, i.e., the second coordinate is strictly bigger than the first coordinate. Create a new path $(w_1, \dots, w_{2n})$ by

$$w_i = \begin{cases} v_i & \text{if } 1 \le i \le r \\ (1,1) - v_i & \text{if } r+1 \le i \le 2n \end{cases}.$$

In the example of a bad path above, $r = 3$ and the new path w is $((1,0),(0,1),(0,1),(0,1))$.
In words, w is the same path as v for the first r steps, but then we swap all of the remaining steps. Geometrically, we are reflecting the rest of the path across the line $y = x + 1$.
Show that $w_1 + \cdots + w_{2n} = (n-1, n+1)$.

(c) In (b) we defined a function from the set of bad paths to the set of paths from $(0,0)$ to $(n-1, n+1)$. Show that this function is a bijection.

(d) Conclude that the number of bad paths is $\binom{2n}{n+1}$, and hence the number of good paths is $\frac{1}{n+1}\binom{2n}{n}$.

(5) Let a, b be real numbers. Give a combinatorial proof of the identity

$$e^{(a+b)x} = e^{ax}e^{bx}.$$

More specifically, show that both sides represent the same formal power series by showing that

$$\sum_{n \ge 0} \frac{((a+b)x)^n}{n!} = \left(\sum_{n \ge 0} \frac{(ax)^n}{n!} \right) \left(\sum_{n \ge 0} \frac{(bx)^n}{n!} \right).$$

Hints:

3: The factorization $1 - x^k + x^{2k} = \frac{1+x^{3k}}{1+x^k}$ might be helpful.

4a: Given a string S of n pairs of parentheses, define $L_i(S)$ to be the number of left parentheses that are in the first i symbols, and define $R_i(S)$ to be the number of right parentheses that are in the first i symbols (so $L_i(S) + R_i(S) = i$ and $L_{2n}(S) = R_{2n}(S) = n$). Prove that S is balanced if and only if $L_i(S) \geq R_i(S)$ for all $1 \leq i \leq 2n$.