Math 184A, Fall 2018
Homework 7
Due: Friday, November 30 by 3:30PM in basement of AP&M

(1) (a) We have n distinguishable telephone polls. We want to paint each one either red, blue, green, or black such that an even number of them is red and an odd number of them is blue. How many ways can this be done?
(b) Continuing with that situation, we add the colors white and yellow, but the total number of polls which are white or yellow must be even. How many ways are there to choose colors?

(2) Let n be a positive integer. Given a group of n people, we want to divide them into nonempty committees and choose a chair and co-chair for each committee (they must be different people), and also choose one of the committees to be in charge of all of the others. Let h_n be the number of ways to do this and set $h_0 = 1$. Give a simple expression for the exponential generating function $H(x) = \sum_{n\geq 0} \frac{h_n}{n!} x^n$.

(3) Recall that a derangement of a set S is a bijection $f\colon S \to S$ such that $f(i) \neq i$ for $i \in S$. (We previously defined this for $S = [n]$ but there's essentially nothing new with this definition.) Let d_n be the number of derangements of $[n]$, and let

$$D(x) = \sum_{n\geq 0} \frac{d_n}{n!} x^n.$$

(a) Without using the formula for d_n that we derived in class, show that

$$D(x)e^x = \frac{1}{1-x}.$$

(b) Dividing both sides by e^x we get $D(x) = e^{-x}\frac{1}{1-x}$. Use the product formula to recover the formula we derived in class:

$$d_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}.$$

(4) Let k be a positive integer. Let $a(k)_n$ be the number of set partitions of $[n]$ into k blocks such that every block has at least 2 elements. Give a simple expression for the exponential generating function

$$A_k(x) = \sum_{n\geq 0} \frac{a(k)_n}{n!} x^n.$$

(5) Let h_n be the number of bijections $f\colon [n] \to [n]$ with the property that $f \circ f \circ f$ is the identity function. Give a simple expression for the exponential generating function $H(x) = \sum_{n\geq 0} \frac{h_n}{n!} x^n$.

Hints:

3: $\frac{1}{1-x}$ is the *exponential* generating function for the sequence $c_n = n!$, i.e., the number of permutations on $[n]$. Use the combinatorial interpretation for products of exponential generating functions.

4: This is like our derivation of the exponential generating function for Stirling numbers. What happens for $k = 1$? $k = 2$?