Math 376, Spring 2018
Homework 2
Due: February 7, 2018 in your discussion section

(1) (Apostol 9.15.8) Find the points on the curve of intersection of the two surfaces
$$x^2 - xy + y^2 - z^2 = 1 \qquad \text{and} \qquad x^2 + y^2 = 1$$
which are closest to the origin.
[You may assume that Lagrange multipliers gives the correct answer in this situation.]

(2) Integrate the vector field $f(x, y) = (xy, xy)$ going counterclockwise along the top half of the unit circle.

(3) (a) Let $\alpha\colon [a, b] \to \mathbf{R}^n$ be a smooth closed loop, i.e., $\alpha(a) = \alpha(b)$. In that case, call $\alpha(a)$ the **base point** of the loop.
We've shown that for a vector field f, the integral $\int f \cdot \mathrm{d}\alpha$ doesn't depend on the parametrization as long as we go in the same direction.
Show that the integral also doesn't depend on which point is the base point.
[Part of this problem is to turn the previous sentence into a precise mathematical statement.]
(b) Let k be a positive integer. In the setup above, let $\alpha^{(k)}$ denote the path which traces out the closed loop α k times in a row. Give a formula for $\alpha^{(k)}$ in terms of α and use it to show that
$$k \int f \cdot \mathrm{d}\alpha = \int f \cdot \mathrm{d}\alpha^{(k)}.$$

(4) Given a scalar field $\varphi\colon \mathbf{R}^n \to \mathbf{R}$ and a piecewise smooth path $\alpha\colon [a, b] \to \mathbf{R}^n$ whose image is C, the **line integral of φ with respect to arc length** is defined to be
$$\int_C \varphi \, \mathrm{d}s = \int_a^b \varphi(\alpha(t)) \sqrt{\alpha_1'(t)^2 + \cdots + \alpha_n'(t)^2} \, \mathrm{d}t.$$
(a) (Apostol 10.9.7) Calculate $\int_C (x + y) \, \mathrm{d}s$ where C is the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ going counterclockwise.
(b) Calculuate $\int_C (2x + 9z) \, \mathrm{d}s$ where C is given by $t \mapsto (t, t^2, t^3)$ for $0 \leq t \leq 1$.

(5) Let $S \subset \mathbf{R}^n$ be a subset. Define a relation $\sim$ on S by $x \sim y$ if there is a continuous path from x to y that stays in S.
(a) Show that $\sim$ is an equivalence relation[1]. The equivalence classes are called the **path-connected components** of S.
(b) If S is open, show that each path-connected component of S is also open.

(6) (Apostol 10.18.18)

[1]This means that: (1) $x \sim x$ for all x, (2) $x \sim y$ implies $y \sim x$, and (3) $x \sim y$ and $y \sim z$ implies that $x \sim z$. An equivalence class is a set of the form $\{x \mid x \sim y\}$ for some fixed y.