Math 376, Spring 2018
Homework 7
Due: March 21, 2018 in your discussion section

(1) Find the surface area of the following surfaces:
  (a) $\{(x, y, z) \mid 3x + 2y + z = 6,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$
  (b) $\{(x, y, z) \mid xy = z,\ x^2 + y^2 \leq 1\}$

(2) Compute $\iint_S F \cdot \mathbf{n}\, \mathrm{d}S$ for the following:
  (a) S is the unit sphere centered at the origin and $F(x, y, z) = (z, y, x)$.
  (b) S is the boundary of the set $\{(x, y, z) \mid 0 \leq z \leq 1 - x^2 - y^2\}$ and $F(x, y, z) = (y, x, z)$.

(3) Let $S \subset \mathbf{R}^3$, let a, b be real numbers, let $F, G\colon S \to \mathbf{R}^3$ be vector fields, and let $\varphi\colon S \to \mathbf{R}$ be a scalar field. Prove the following properties (assume all of the relevant derivatives exist and are continuous):

$$\begin{aligned}
\operatorname{div}(aF + bG) &= a \operatorname{div} F + b \operatorname{div} G \\
\operatorname{curl}(aF + bG) &= a \operatorname{curl} F + b \operatorname{curl} G \\
\operatorname{div}(\operatorname{curl} F) &= 0 \\
\operatorname{div}(\varphi F) &= \varphi \operatorname{div} F + \nabla\varphi \cdot F \\
\operatorname{curl}(\varphi F) &= \varphi \operatorname{curl} F + \nabla\varphi \times F.
\end{aligned}$$

(4) Let μ be a scalar field which is always nonzero and F a vector field (both defined on some region in $\mathbf{R}^3$) such that μF is a gradient. Prove that F is perpindicular to $\operatorname{curl} F$. Again, just assume all relevant derivatives exist and are continuous.

(5) Compute $\iint_S \operatorname{curl}\, F \cdot \mathbf{n}\, \mathrm{d}S$ where S is the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 \leq 1$.

(6) Let $S \subseteq \mathbf{R}^3$ be the 2-dimensional sphere of radius R centered at the origin. Since S has no boundary, Stokes' theorem tells us that $\iint_S \operatorname{curl}\, F \cdot \mathbf{n}\, \mathrm{d}S = 0$ for any F since it turns into an integral over the empty set. This might feel strange, so verify this independently using Stokes' theorem by showing that

$$\iint_{S_1} \operatorname{curl}\, F \cdot \mathbf{n}\, \mathrm{d}S = -\iint_{S_2} \operatorname{curl}\, F \cdot \mathbf{n}\, \mathrm{d}S$$

where S_1 is the $z \geq 0$ part of the sphere and S_2 is the $z \leq 0$ part of the sphere.