Math 184, Fall 2019
Homework 2
Due: Friday, Oct. 18 by 3:00PM in homework box #2 in basement of AP&M
(late homework will not be accepted)

Explanations should be given for your solutions. Use complete sentences. Some hints are on the last page.

(1) Consider the equation

$$x_1 + x_2 + \cdots + x_6 = 30.$$

For each of the following conditions, how many solutions are there? (Each part is an independent problem, don't combine the conditions.)

(a) The x_i are non-negative even integers.
(b) The x_i are non-negative odd integers.
(c) The x_i are non-negative integers and $x_6 \le 2$.

(2) We consider some variations of standard Poker hands. Start with a standard deck of cards (4 suits, 13 values, so 52 cards in total). You will count ways to choose 6 cards.

(a) How many ways can we have two triples? i.e., 3 of the cards have the same value and the other 3 cards also have the same value.
(b) How many ways can we have exactly two pairs? i.e., 2 of the cards have the same value, another 2 cards have the same value (but different from the first), and the remaining 2 cards have different values from these cards and each other.
(c) An "alternating straight" is a choice of 6 cards whose values can be put in consecutive order, and the suits alternate between two different suits. An example is $5\heartsuit, 6\diamondsuit, 7\heartsuit, 8\diamondsuit, 9\heartsuit, 10\diamondsuit$. How many are there?

(3) Find a simple formula for $S(n, n-2)$, i.e., the number of partitions of $[n]$ into $n-2$ blocks (assume that $n \ge 3$).

(4) Let $F(n)$ be the number of all partitions of $[n]$ such that every block has size ≥ 2. Prove that

$$B(n) = F(n) + F(n+1),$$

where $B(n)$ is the nth Bell number.

(5) Fix an integer $n \ge 2$. Call a composition $(a_1, \dots, a_k)$ of n **doubly even** if the number of a_i which are even is also even (i.e., there could be no even a_i, or 2 of them, or 4, or ...).

Show that the number of doubly even compositions of n is 2^{n-2}.

For example, if $n = 4$, then here are the 4 doubly even compositions of 4:

$$(2,2), \qquad (3,1), \qquad (1,3), \qquad (1,1,1,1).$$

Hints

(1) (a) Since x_i is even, write it as $x_i = 2y_i$.
(b) Same, but write $x_i = 2y_i + 1$.

(4) By definition, $B(n) - F(n)$ is the number of partitions of $[n]$ such that there is at least 1 block with size 1. Show that this number is $F(n+1)$ as follows: given such a partition, add a new singleton $\{n+1\}$ and then merge together all of the singleton blocks into a single block.

(5) Given a composition $\alpha = (a_1, \dots, a_k)$, define another composition $\Phi(\alpha)$ by

$$\Phi(\alpha) = \begin{cases} (1, a_1 - 1, a_2, a_3, \dots, a_k) & \text{if } a_1 > 1 \\ (a_2 + 1, a_3, \dots, a_k) & \text{if } a_1 = 1 \end{cases}.$$

(in both cases, we didn't do anything to $a_3, \dots, a_k$). Show that Φ defines a bijection between the set of doubly even compositions of n and the set of compositions of n which are not doubly even.