Math 184, Fall 2019
Homework 5
Due: Friday, Nov. 15 by 3:00PM in homework box #2 in basement of AP&M
(late homework will not be accepted)

Explanations should be given for your solutions. Use complete sentences. Some hints are at the very end.

(1) For $n > 0$, let $a_n$ be the number of partitions of $n$ such that every part appears at most twice, and let $b_n$ be the number of partitions of $n$ such that no part is divisible by 3. Set $a_0 = b_0 = 1$. Show that $a_n = b_n$ for all $n$.

(2) Let $a_n$ be the sequence which satisfies the recurrence

$$a_n = a_{n-1} + 2\sum_{i=0}^{n-2} a_i a_{n-2-i} \qquad \text{for } n \geq 2$$

and $a_0 = a_1 = 1$. Find a simple formula for $A(x) = \sum_{n\geq 0} a_n x^n$.

(3) A forward path from $(0, 0)$ to $(n, n)$ is **good** if it never goes strictly above the diagonal line $x = y$. Any other forward path is **bad**. From class, the number of good forward paths is the $n$th Catalan number. In this problem, you will get a new derivation for the formula for Catalan numbers without using generating functions. We denote paths as sequences $(v_1, \dots, v_{2n})$ where each $v_i$ is either the vector $(1, 0)$ or $(0, 1)$.

(a) Given a bad path $(v_1, \dots, v_{2n})$ from $(0, 0)$ to $(n, n)$, let $r$ be the smallest index such that $v_1 + \cdots + v_r$ is above the line $x = y$, i.e., the second coordinate is strictly bigger than the first coordinate. Create a new path $(w_1, \dots, w_{2n})$ by

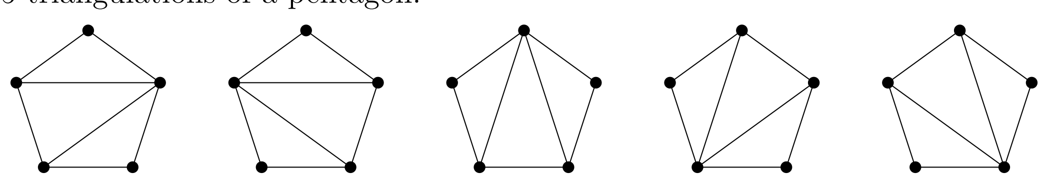
$$w_i = \begin{cases} v_i & \text{if } 1 \leq i \leq r \\ (1,1) - v_i & \text{if } r+1 \leq i \leq 2n \end{cases},$$

i.e., $w$ agrees with $v$ for the first $r$ steps, and we swap all of the remaining steps. Show that $w$ is a forward path from $(0, 0)$ to $(n - 1, n + 1)$.

(b) In (a) we defined a function {bad forward paths from $(0, 0)$ to $(n, n)$} $\to$ {forward paths from $(0, 0)$ to $(n - 1, n + 1)$}. Show that this function is a bijection.

(c) Use the formula in HW3, #5(a) and the previous part to get a formula for the number of good forward paths from $(0, 0)$ to $(n, n)$.

(4) Let $n$ be a positive integer. Show that the number of ways of triangulating (i.e., drawing diagonals between vertices that do not intersect except at vertices so that the regions are all triangles) a convex polygon with $(n + 2)$ vertices is the Catalan number $C_n$. By convention, the 2-gon has exactly one triangulation and here are the 5 triangulations of a pentagon:



(5) There are $n$ aisles of shelves in a store. We want to separate them into consecutive nonempty groups for different categories of items. In addition, each category will be painted either red, blue, or green, and we will select some nonempty subset of the categories to be featured in the weekly advertisement. Let $h_n$ be the number of ways to do this. Express $H(x) = \sum_{n\geq 0} h_n x^n$ as a rational function.

Hints:

1: The identity $1 + x^i + x^{2i} = \dfrac{1 - x^{3i}}{1 - x^i}$ may be useful.

4: Fix a vertex $v$ and consider the first vertex (going counterclockwise) that shares a diagonal with it. If none exists, then pick the vertex immediately clockwise of $v$. This splits the polygon into a left part (where $v$ is no longer on a diagonal) and a right part (which is empty in the exceptional case).