Math 184A, Final exam
Instructor: Steven Sam
December 13, 2018

Name:

Discussion time (circle): 4PM 5PM 7PM 8PM

Also write your name on the back of the last page.

Problem	Score
1	/ 10
2	/ 10
3	/ 12
4	/ 9
5	/ 8
6	/ 11
Total	/ 60

- No books, materials, notes, cell phones, calculators, etc. are allowed during the exam.
- Cross out or erase irrelevant scratch work. If you write incorrect statements without crossing them out, you may lose points. **It must be clear what your final answer is.**
- When asked to explain or prove, give enough detail so that we know that you are not guessing the answer. We are not mind readers, so you will not receive the benefit of the doubt if you skip too much detail.
- If you need more space, you may use the backs of the pages and also there is a blank sheet at the end. Please clearly indicate which problem you are working on. If you still need more paper, raise your hand.

Good luck!

1. (3+3+4 points) You don't need to explain your answer, but a wrong answer with no explanation might receive 0 points.

   (a) How many compositions of 15 into 8 parts are there?

   (b) What is the coefficient of $w^3x^3y^2z^3$ in $(w+x+y+z)^{11}$?

   (c) How many paths are there from $(0,0,0)$ to $(3,2,4)$ using only the steps $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$?

2. (5+5 points) Explain your answer or show your work:

   (a) How many necklaces are there of length 27 using k different colors for the beads?

   (b) We paint n distinguishable chairs either blue, white, or yellow such that the total number which are blue or white is odd. How many ways can this be done?

3. (6+6 points) Explain your answer or show your work:

(a) How many integers $1 \leq x \leq 30000$ are not divisible by 2, 3, or 5?

(b) Let a_n be the number of set partitions of $[n]$ such that every block has size ≥ 3. By convention $a_0 = 1$. Give a simple expression for $A(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n$.

4. (4+5 points) Explain your answer or show your work:

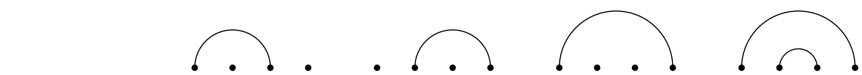
(a) Let $A(x) = \sum_{n\geq 0} a_n x^n$ be a formal power series. Express $B(x) = \sum_{n\geq 0} b_n x^n$ in terms of $A(x)$ if $b_n = \begin{cases} a_n + 3a_{n-1} & \text{if } n \geq 1 \\ a_0 & \text{if } n = 0 \end{cases}$.

(b) $C(x) = \sum_{n\geq 0} c_n x^n$ is a formal power series satisfying the relation $C(x) = \dfrac{x}{(1 - 3C(x))^2}$. Find a formula for c_n.

5. (8 points) Let a, m, n be positive integers. Prove that the following identity is correct:

$$\binom{n+m}{a} = \sum_{i=0}^{a} \binom{n}{i} \binom{m}{a-i}.$$

6. (6+5 points) Draw n dots in a line. An **arc diagram** is a way to draw arcs (possibly none) that join some of the dots so that no two arcs intersect or share a dot. When $n = 4$, here are all of the arc diagrams:



Let m_n be the number of arc diagrams (by convention, $m_0 = m_1 = 1$).

(a) Prove that $m_n = m_{n-1} + \sum_{i=0}^{n-2} m_i m_{n-2-i}$ for $n \geq 2$.

The formula from (a) is $m_n = m_{n-1} + \sum_{i=0}^{n-2} m_i m_{n-2-i}$. You may use it for (b) even if you did not solve (a).

(b) Find a simple expression for $M(x) = \sum\limits_{n \geq 0} m_n x^n$.

Space for extra scratch work