Math 251C, Spring 2020
Homework 3

(1) Let R be a commutative ring. A sequence $f_1, \dots, f_r \in R$ is a **regular sequence** if:
   - For all i, multiplication by f_i on $R/(f_1, \dots, f_{i-1})$ is injective, i.e., $gf_i \in (f_1, \dots, f_{i-1})$ implies that $g \in (f_1, \dots, f_{i-1})$ (for $i = 1$, we interpret $(f_1, \dots, f_{i-1}) = 0$).
   - $(f_1, \dots, f_r) \neq R$

   (a) Now suppose $R = \mathbf{k}[x_1, \dots, x_n]$ is a polynomial ring over a field $\mathbf{k}$ and that each f_i is a homogeneous polynomial of degree d_i. Show that
   $$\sum_{d \geq 0} \dim_{\mathbf{k}}(R/(f_1, \dots, f_r))_d t^d = \frac{\prod_{i=1}^r (1 - t^{d_i})}{(1-t)^n}$$
   where the subscript denotes the space of degree d homogeneous elements.

   (b) Prove Theorem 2.3.2 (first prove the map is surjective without the regularity assumption, then use dimension counting in each degree to prove injectivity).

(2) Prove Corollary 2.3.3 using Theorem 2.3.2.

(3) Prove Proposition 2.3.4 using Theorem 2.3.5.

(4) This exercise gives generators for $\mathbf{SO}_m(\mathbf{C})$ and outlines a proof that it is connected in the Zariski topology (it easily applies also to the standard Euclidean topology).

   (a) Let V be an orthogonal space with orthogonal form β and pick a non-isotropic $a \in V$. Define
   $$s_a(x) = x - \frac{2\beta(x, a)}{\beta(a, a)} a.$$
   Show that $s_a \in \mathbf{O}(V)$ and $\det(s_a) = -1$. This is called an **(orthogonal) reflection**.

   (b) The Cartan–Dieudonné theorem states: every element $g \in \mathbf{O}(V)$ is a product of $\leq \dim V$ many reflections. Prove it as follows.

      (i) If $\dim V \leq 2$, do it directly.
      Otherwise, $\dim V \geq 3$ and we split it into 3 cases.

      (ii) Case 1: g fixes a non-isotropic vector v, (use that g fixes $v^\perp$).

      (iii) Case 2: There is a non-isotropic vector v such that $v - g(v)$ is non-isotropic. Show that $s_{v-g(v)}g$ fixes v and appeal to Case 1.

      (iv) Case 3: Every fixed point of g is isotropic and for every non-isotropic v, $v - g(v)$ is isotropic. In that case, prove that
      - $v - g(v)$ is isotropic for all $v \in V$.
        Hint: pick a non-isotropic vector $w \in v^\perp$ and use that $\pm v - w$ are non-isotropic.
      - $(1 - g)^2 = 0$ and hence $\det(g) = 1$.
        Hint: Use that $\beta(v - g(v), v - g(v)) = 0$ for all v implies that $\beta(v - g(v), w - g(v)) = 0$ for all v.
      - $\dim V$ is even.
        Hint: Both the image and kernel of $1 - g$ are isotropic subspaces and their dimensions add up to $\dim V$.

      To finish: for any $w \in V$, $\det(s_w g) = -1$ so $s_w g$ must be in either Cases 1 or 2, and hence is a product of $\leq \dim V$ reflections. However, since $\dim V$ is even, it must actually be a product of $\leq \dim V - 1$ reflections.

(c) We show that $\mathbf{SO}_m(\mathbf{C})$ is connected by constructing a path from the identity to any g. It suffices to do it when $g = s_b s_a$ for non-isotropic vectors a, b, since by the last part, every element in $\mathbf{SO}_m(\mathbf{C})$ is a product of an even number of reflections.
Construct a polynomial function $\varphi\colon \mathbf{C} \to \mathbf{C}^m$ such that $\varphi(t)$ is non-isotropic for all t and which contains both a and b in its image. Then the desired path is $\alpha_g\colon \mathbf{C} \to \mathbf{SO}_m(\mathbf{C})$ given by $\alpha_g(t) = s_b s_{\varphi(t)}$.

(5) Prove the following properties about isotropic subspaces with respect to an orthogonal form:
   (a) If V is isotropic, then $\dim V \le n$.
   (b) Given 2 isotropic subspaces V_1, V_2 with $\dim V_1 = \dim V_2$, there exists $g \in \mathbf{O}_m(\mathbf{C})$ such that $gV_1 = V_2$. Assuming that either m is odd, or that m is even and $\dim V_i < n$, we can actually find $g \in \mathbf{SO}_m(\mathbf{C})$ such that $gV_1 = V_2$. In the exceptional case that m is even and $\dim V_i = n$, there are 2 orbits of isotropic subspaces under the action of $\mathbf{SO}_{2n}(\mathbf{C})$. In particular, the span of $e_1, \dots, e_n$ and $e_1, \dots, e_{n-1}, e_{n+1}$ are in separate orbits.
   (c) Every isotropic subspace is contained in an n-dimensional isotropic subspace.

(6) Prove the Newell–Littlewood product formula for the orthogonal group.