Math 251C, Lecture1

Note Title 3/30/2020

<u>Goal</u>: Representations of Classical Groups / $\mathbb{C}$

3 Series:

- general linear groups $GL_n(\mathbb{C})$
- symplectic groups $Sp_{2n}(\mathbb{C})$
- orthogonal groups $O_n(\mathbb{C})$

<u>Definitions</u>

V = finite-dim'l complex vector space

$GL(V)$ = group of invertible linear operators on V

general linear group

$= \{ g : V \longrightarrow V \mid g \text{ linear, invertible} \}$

group operation is composition

If $V = \mathbb{C}^n$, we write $GL_n\mathbb{C}$ instead of $GL(V)$

(This is a complex Lie group)

Representations

Notation:

$B \subset GL_n\mathbb{C}$ is subgroup of upper-triangular matrices (Borel subgroup)

$T \subset B$ is subgroup of all diagonal matrices (maximal torus)

For $GL(V)$, a Borel subgroup is any subgroup which is the upper-triangular matrices for some choice of basis

Similarly, a maximal torus is any subgroup which is the diagonal matrices for some choice of basis

<u>Def</u>. An <u>algebraic (rational) representation</u> of $GL(V)$ is a group homomorphism $\rho : GL(V) \longrightarrow GL(W)$ for some f.d. complex vector space W s.t.: for some (and hence, any) choice of bases for V & W, entries of $\rho(g)$ are rational functions of entries of $g \in GL(V)$

ρ is <u>polynomial</u> if these functions are polynomial.

<u>Examples</u> ① $V = W$, $\rho = id$.

② $V = W = \mathbb{C}^n$, $\rho(g) = (g^{-1})^T$

cofactor formula for inverse shows entries of g^{-1} are rational functions of entries of g

Not polynomial

③ $V = \mathbb{C}^2$, $W = \mathbb{C}^3$

$$\rho : GL_2\mathbb{C} \longrightarrow GL_3\mathbb{C}$$

$$\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

Polynomial representation

④ $V = \mathbb{C}^n$ $W = \mathbb{C}$ $\rho(g) = (\det g)^d, d \in \mathbb{Z}$

Polynomial iff $d \geq 0$

· NON-EXAMPLE

$V = W = \mathbb{C}^n$, $\rho(g) = \overline{g}$

complex conjugate each entry

If $n=1$, $\bar{z} = a - bi$ is not a polynomial of $a+bi = z$

Alternatively, ρ gives a group action of G on W, i.e., $g \cdot w := \rho(g)(w)$ (plus the algebraic condition)

Usually, we just say representation to mean algebraic rep.

And usually use W to denote the representation, not ρ.

Review of general facts

Pick a maximal torus $T \subset GL(V)$.

Let $\rho : GL(V) \to GL(W)$ be a rep.

Fact 1. $\exists$ a basis $w_1, w_2, \ldots, w_r$ of W s.t. they are eigenvectors for $\rho(t)$ $\forall t \in T$

$\Longrightarrow$ wrt this basis, $\rho(t)$ is diagonal $\forall t \in T$

Say $t = \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & \ddots \\ & & & x_n \end{pmatrix}$. Then the eigenvalues of each w_i are of the form $x_1^{\mu_{i,1}} x_2^{\mu_{i,2}} \cdots x_n^{\mu_{i,n}}$ for some $(\mu_{i,1}, \mu_{i,2}, \ldots, \mu_{i,n}) \in \mathbb{Z}^n$

weight of w_i

w_i is called weight vector

For weight $\mu \in \mathbb{Z}^n$, $\mu(t) := x_1^{\mu_1} \cdots x_n^{\mu_n}$

where $t = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}$

<u>Def</u>. The <u>character</u> of ρ (or W) is the function $(\operatorname{char} \rho)(x_1, \ldots, x_n) =$

$$\operatorname{Trace}\left(\rho\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}\right) = \sum_{i=1}^{r} x_1^{\mu_{i,1}} \cdots x_n^{\mu_{i,n}}$$

Examples (from last time)

(1) Pick standard basis $e_1, \dots, e_n$ for $V = W$. Then $\rho\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$

The e_i are eigenvectors for $\rho(t)$.

weight of e_i is $(0, \dots, \underset{\substack{\uparrow \\ i}}{1}, \dots, 0)$

since $\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} e_i = x_i e_i$

$(\operatorname{char}\rho)(x_1, \dots x_n) = x_1 + x_2 + \dots + x_n$.

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(2) Pick standard basis $e_1 \dots e_n$ for $V = W$.

$\rho\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} = \begin{pmatrix} x_1^{-1} & & 0 \\ & \ddots & \\ 0 & & x_n^{-1} \end{pmatrix}$ e_i's are eigenvectors for $\rho(t)$

weight of e_i is $(0, \dots, \underset{\substack{\uparrow \\ i}}{-1}, \dots 0)$ since

$\begin{pmatrix} x_1^{-1} & & \\ & \ddots & \\ & & x_n^{-1} \end{pmatrix} e_i = x_i^{-1} e_i$.

$(\operatorname{char}\rho)(x_1, \dots, x_n) = x_1^{-1} + x_2^{-1} + \dots + x_n^{-1}$

③ Pick standard bases for both $\mathbb{C}^2, \mathbb{C}^3$

$$\rho\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 & 0 & 0 \\ 0 & x_1 x_2 & 0 \\ 0 & 0 & x_2^2 \end{pmatrix}$$

e_1, e_2, e_3 are eigenvectors for $\rho(t)$ $\forall t \in T$

weight of $e_1 = (2,0)$

weight of $e_2 = (1,1)$

weight of $e_3 = (0,2)$

$(\mathrm{char}\ \rho)(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$.

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④ Pick standard bases for $\mathbb{C}^n, \mathbb{C}$.

$$\rho\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} = (x_1 \cdots x_n)^d$$

$\underset{\mathbb{C}}{\overset{\Uparrow}{e_1}}$ is eigenvector for $\rho(t)$ w/ weight $(d, d, \ldots, d)$

$(\mathrm{char}\ \rho)(x_1, \ldots x_n) = x_1^d x_2^d \cdots x_n^d$.

Basic operations.

① Given reps $\rho_1 : GL_n\mathbb{C} \to GL(W_1)$

$\rho_2 : GL_n\mathbb{C} \to GL(W_2)$

$\Longrightarrow$ direct sum representation

$$\rho = \rho_1 \oplus \rho_2 : GL_n\mathbb{C} \longrightarrow GL(W_1 \oplus W_2)$$

In matrices, $\rho(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$

In terms of group action,

$$g \cdot (w_1, w_2) = (g \cdot w_1, g \cdot w_2)$$

$$\forall g \in GL_n\mathbb{C},\ w_1 \in W_1,\ w_2 \in W_2.$$

$$(\mathrm{char}\,\rho_1 \oplus \rho_2)(x_1, \ldots x_n) = (\mathrm{char}\,\rho_1)(x_1, \ldots, x_n) + (\mathrm{char}\,\rho_2)(x_1, \ldots x_n)$$