Math 251C, Lecture 11

Note Title 4/22/2020

Multiplicity-free action

U = space of $2n \times n$ matrices

$GL_n\mathbb{C} \times Sp_{2n}\mathbb{C}$ acts on U by

$$(g,h)\cdot u = (h^{-1})^T u g^{-1}$$

Define $X = \{u \in U \mid u^T \Omega u = 0\}$

Note: $u^T\Omega u$ is a skew-symmetric matrix whose entries are quadratic polynomials in entries of u

$\Rightarrow$ X = affine variety defined by $\binom{n}{2}$ quadratic poly

Alternatively, $X = \{u : \mathbb{C}^n \to \mathbb{C}^{2n} \mid u(\mathbb{C}^n) \text{ is isotropic}\}$

X is closed under action of $GL_n\mathbb{C} \times Sp_{2n}\mathbb{C}$

Prop. X is irreducible.

Pf. X° = matrices in X of full rank

Show: ① X° has a transitive action under $GL_n\mathbb{C} \times Sp_{2n}\mathbb{C}$

② X° is dense in X

① : If $u \in X^\circ$, $u(\mathbb{C}^n)$ is a Lagrangian, and $Sp_{2n}\mathbb{C}$ acts transitively on set of Lagrangians

Given, $u, u' \in X^\circ$ whose images are equal, I can precompose u by element $g \in GL_n\mathbb{C}$ so that $ug = u'$.

$\Longrightarrow$ Pick a point $x \in X^\circ$, define map

$$GL_n\mathbb{C} \times Sp_{2n}\mathbb{C} \longrightarrow X^\circ \quad \text{which is surjective}$$
$$(g,h) \longmapsto (g,h)\cdot x$$

From last time, $GL_n\mathbb{C}$, $Sp_{2n}\mathbb{C}$ are connected

$\overset{HW}{\Longrightarrow}$ they are irreducible

$\Longrightarrow$ X° is irreducible

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② Let $Y \subseteq X$ be a closed subset that contains X°

Need to show that $Y = X$.

Assume not, so $Y \neq X$, pick $u \in X \setminus Y$.

Since $u \notin X^\circ$, $\operatorname{rk} u = r < n$. We can find a basis $v_1, \dots, v_n$ for $\mathbb{C}^n$ so that $u(v_1), \dots, u(v_r)$ are linearly independent, and $u(v_{r+1}) = \dots = u(v_n) = 0$.

$u(v_1), \dots, u(v_r)$ span isotropic space, so I can find $w_{r+1}, \dots, w_n \in \mathbb{C}^{2n}$ so that $u(v_1), \dots u(v_r), w_{r+1}, \dots w_n$ span a Lagrangian.

Define $\varphi : \mathbb{C} \longrightarrow X$ by letting $\varphi(t) \in X$ the linear map $\varphi(t)(v_i) = u(v_i)$ $i = 1, \dots, r$

$\varphi(t)(v_i) = t w_i$ $i = r+1, \dots n$

φ is continuous (in Zariski topology)

$\varphi(0) = u$ and $\varphi(t) \in X^\circ$ if $t \neq 0$.

Since Y closed $\Rightarrow$ $\underset{\underset{\mathbb{C} \setminus 0}{\shortparallel}}{\varphi^{-1}(Y)} \subseteq \mathbb{C}$ closed

$\Rightarrow$ $\mathbb{C} = (\mathbb{C} \setminus 0) \cup 0$, both closed

contradicts that $\mathbb{C}$ is irreducible.

$\Rightarrow$ u does not exist, so $Y = X$

$\Rightarrow$ X° dense.

If $X = X_1 \cup X_2$ (closed) $\Rightarrow$ $X^\circ = (X^\circ \cap X_1) \cup (X^\circ \cap X_2)$

$\Rightarrow$ $X^\circ = X^\circ \cap X_i$ for some i $\Rightarrow$ $X^\circ \subseteq X_i$

By density, $X_i = X$. $\longrightarrow$ X is irreducible $\square$

$J = 2n \times n$ matrix $\in X$

$J_{i,i} = 1$ for $i = 1, \ldots n$, 0 otherwise

$B \subset GL_n \mathbb{C}$, $B' \subseteq Sp_{2n} \mathbb{C}$ upper-triangular matrices

$f_i \in \mathbb{C}[X]$ is determinant of upper left $i \times i$ submatrix.

<u>Prop</u> f_i is a highest weight vector of weight $(\underbrace{1, \ldots, 1}_{i}, 0, \ldots 0)$, $(\underbrace{1, \ldots, 1}_{i}, 0, \ldots 0)$

<u>Lemma</u>. The $B \times B'$ orbit Y of J is open, dense.

<u>Pf</u>. <u>Claim</u>: $Y = \{A \in X \mid f_i(A) \neq 0 \text{ for } i = 1, \ldots, n\}$

Idea for $\supseteq$: use row/column operations on A to get to J

GL_n action: can add any multiple of a column of A to any column appearing to the right of it, to turn top half of A into lower-triangular

Sp_{2n} action: zero out lower-triangular entries and bottom half of A $\square$

<u>Lemma</u>. If (λ, λ') is a weight of a h.w. vector in $\mathbb{C}[X]$ then $\lambda = \lambda'$.

<u>Pf</u>. stab(J) contains $\begin{pmatrix} x_1^{-1} & & 0 \\ & \ddots & \\ 0 & & x_n^{-1} \end{pmatrix} \begin{pmatrix} x_1 & & & & & 0 \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_n^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & x_1^{-1} \end{pmatrix}$

If (λ, λ') appears as h.w., then

$$x_1^{\lambda_1' - \lambda_1} x_2^{\lambda_2' - \lambda_2} \cdots x_n^{\lambda_n' - \lambda_n} = 1 \quad \forall x_1 \ldots x_n \in \mathbb{C}^*$$

$\Rightarrow$ all exponents are 0. $\square$

We can realize all of these highest weights by taking products of the $f_1, \ldots f_n$.

<u>Cor</u> (Symplectic Cauchy identity)

$$\mathbb{C}[X] \cong \bigoplus_{\lambda} S_\lambda \mathbb{C}^n \otimes S_{[\lambda]} \mathbb{C}^{2n} \text{ as}$$

$\lambda \leftarrow \lambda_1 \geq \cdots \geq \lambda_n \geq 0$

representations of $GL_n\mathbb{C} \times Sp_{2n}\mathbb{C}$

Branching problem

How does $S_\lambda \mathbb{C}^{2n}$ decompose as $Sp_{2n}\mathbb{C}$-rep?

We're going to study using X.

Prop. $\dim X = \dfrac{3n^2+n}{2} = \dim U - \binom{n}{2}$

Pf. First, describe $stab(J)$ in $GL_n\mathbb{C} \times Sp_{2n}\mathbb{C}$

If $(g,h) \in stab(J) \implies h^T J g = J$. Write

$h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}$ all $n \times n$. $\rightsquigarrow$ $h_1 = (g^{-1})^T$, $h_2 = 0$

$h \in Sp_{2n}\mathbb{C} \implies h^T \Omega h = \Omega$

$h_4 = I' g I'$ and $-h_3^T I' g^{-1} + (g^{-1})^T I' h_3 = 0$

$\implies h_3 g$ symmetric wrt antidiagonal.

An element of $stab(J)$: pick $g \in GL_n\mathbb{C}$ freely $\to n^2$ dim

h_1, h_2, h_4 determined

$h_3 \rightsquigarrow \binom{n+1}{2}$ dim $\implies \dim stab(J) = n^2 + \binom{n+1}{2}$

$\dim X = \dim$ orbit of $J = \dim GL_n\mathbb{C} + \dim Sp_{2n}\mathbb{C} - \dim stab J$ $\square$