Math 251C, Lecture 15

Note Title 5/1/2020

$\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} = I' \in \mathrm{Sym}^2 \mathbb{C}^m$, $O_m\mathbb{C} = \{g \mid g^T I' g = I'\}$

$SO_m\mathbb{C} = \{g \in O_m\mathbb{C} \mid \det g = 1\}$

$\omega(u,v) = u^T I' v$

basis $f_1 \dots f_m$ <u>hyperbolic</u> if $\omega(f_i, f_{m+1-j}) = \delta_{ij}$

Borel = uppertriangular in $SO_m\mathbb{C}$ wrt hyperbolic basis

max torus = diagonal in $SO_m\mathbb{C}$ wrt hyperbolic basis

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Given rep $\rho: SO_m\mathbb{C} \to GL(W)$,

$(\mathrm{char}\,\rho)(x_1, \dots, x_n) = \mathrm{trace}\ \rho\begin{pmatrix} x_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & x_n & & & & \\ & & & (1?) & & & \\ & & & & x_n^{-1} & & \\ & & & & & \ddots & \\ 0 & & & & & & x_1^{-1} \end{pmatrix}$

$m = 2n + \varepsilon$

$\varepsilon \in \{0, 1\}$

<u>Thm.</u> Representations of $SO_m\mathbb{C}$ determined by character

Irred reps $\longleftrightarrow$ dominant weights

a weight $\mu = (\mu_1, \dots, \mu_n)$ is dominant if:

m odd: $\mu_1 \geq \dots \geq \mu_n \geq 0 \in \mathbb{Z}^n$

m even: $\mu_1 \geq \dots \geq \mu_{n-1} \geq |\mu_n| \in \mathbb{Z}^n$

If m is odd, let $S_{[\mu]}\mathbb{C}^m$ = irred rep w/ hw μ

If m is even and $\mu_n = 0$, do same.

If $\mu_n > 0$

$$\begin{cases} \text{let } S_{[\mu]^+}\mathbb{C}^m = \text{irred rep w/ h.w. } \mu \\ S_{[\mu]^-}\mathbb{C}^m = \text{irred rep w/ h.w. } (\mu_1, \ldots, \mu_{n-1}, -\mu_n) \\ S_{[\mu]}\mathbb{C}^m = S_{[\mu]^+}\mathbb{C}^m \oplus S_{[\mu]^-}\mathbb{C}^m \end{cases}$$

Question. How does $S_\lambda \mathbb{C}^m$ decompose as sum of irreducible SO_m-reps?

Prop. $SO_m\mathbb{C}$ is connected (in Zariski topology).

Note. $O_m\mathbb{C}$ is disconnected:

$\det : O_m\mathbb{C} \longrightarrow \mathbb{C}$ continuous, but image is ± 1 which is disconnected.

It has 2 connected components: $SO_m\mathbb{C}$, det -1 matrices.

$SO_m\mathbb{C} \subset O_m\mathbb{C}$ normal subgroup of index 2, w/ quotient $\mathbb{Z}/2$

If m is odd, $-Id \in O_m\mathbb{C}$ and is central

$\Rightarrow O_m\mathbb{C} \cong SO_m\mathbb{C} \times \mathbb{Z}/2$

If m is even, $O_m\mathbb{C}$ not a direct product of $SO_m\mathbb{C}$ and $\mathbb{Z}/2$

Rep theory of $O_{2n}\mathbb{C}$ summary:

λ partition w/ $\ell(\lambda) \leq n-1$, then we get pair of irreps V_λ, $V_\lambda \otimes \det$ of $O_{2n}\mathbb{C}$ whose restriction to $SO_{2n}\mathbb{C}$ is $S_{[\lambda]}\mathbb{C}^{2n}$.

If $\ell(\lambda) = n$, then there is irrep V_λ of $O_{2n}\mathbb{C}$ s.t. $V_\lambda \cong V_\lambda \otimes \det$ whose restriction to $SO_{2n}\mathbb{C}$ is $S_{[\lambda]}\mathbb{C}^{2n} = S_{[\lambda]^+}\mathbb{C}^{2n} \oplus S_{[\lambda]^-}\mathbb{C}^{2n}$.

<u>Rmk</u>. Reps of Lie groups $\xrightarrow{\text{diff.}}$ reps of the Lie algebra

integrate (sometimes)

Can integrate for simply-connected form of our group.

GL_n, Sp_{2n} simply-connected

SO_m is <u>not</u> simply-connected, $\pi_1(SO_m\mathbb{C}) \cong \mathbb{Z}/2$

$\Rightarrow \exists$ double cover $Spin_m\mathbb{C} \longrightarrow SO_m\mathbb{C}$

Multiplicity-free action

ω orthogonal form on $\mathbb{C}^m$ $\quad (m = 2n+\varepsilon,\ \varepsilon \in \{0,1\})$

A subspace $V \subset \mathbb{C}^m$ is <u>isotropic</u> if $\omega(u,v) = 0$ $\forall u, v \in V$

Properties:

① If V isotropic $\Longrightarrow \dim V \leq n$

② Given isotropic subspaces V_1, V_2 of same dim, $\exists\, g \in O_m\mathbb{C}$ s.t. $gV_1 = V_2$.

If m is odd, or m is even and $\dim V_i < n$, then $\exists\, g \in SO_m\mathbb{C}$ s.t. $gV_1 = V_2$

If $m = 2n$, $\dim V_i = n$, there are 2 orbits under $SO_{2n}\mathbb{C}$. $\operatorname{Span}\langle e_1, \dots e_n\rangle$, $\operatorname{Span}\langle e_1, \dots e_{n-1}, e_{n+1}\rangle$ are in different orbits.

③ Every isotropic subspace is contained in an n-dim isotropic subspace.

$U = m \times n$ matrices. Define action of $GL_n\mathbb{C} \times O_m\mathbb{C}$ on U by: $(g,h) \cdot u = (h^{-1})^T u g^{-1}$.

Define $X = \{u \in U \mid u^T I' u = 0\}$
$= \{u : \mathbb{C}^n \to \mathbb{C}^m \mid u(\mathbb{C}^n) \text{ is isotropic}\}$

symmetric matrix whose entries are deg 2 polynomials in U

X is affine variety defined by $\binom{n+1}{2}$ polynomials

X is closed under action of $GL_n\mathbb{C} \times O_m\mathbb{C}$.

<u>Prop</u>. X is irreducible if m is odd

X has 2 irreducible components if m is even.

<u>Pf</u>. Consider action of $GL_n \times SO_m\mathbb{C}$ (connected) on full rank matrices. It has 1 orbit if m odd and 2 orbits if m even. Taking closure of these orbits gives irreducible components. $\square$

$J \in X$ $J_{i,i} = 1$ for $i = 1, \dots, n$, 0 otherwise

If m even, $J' \in X$ $J_{i,i} = 1$ for $i = 1, \dots, n-1$, 0 else.
$J_{n+1,n} = 1$

$B \subset GL_n \mathbb{C}$, $B' \subset SO_m \mathbb{C}$ Borel subgroups.

$f_i \in \mathbb{C}[X]$, $f_i(u) = \det$ of upper left $i \times i$ submatrix.

If m even, $f_n' \in \mathbb{C}[X]$, $f_n'(u) = \det$ of $n \times n$ submatrix w/ rows $1, \dots, n-1, n+1$

Prop. f_i is a h.w. vector of h.w. $(1^i, 0^{n-i}), (1^i, 0^{n-i})$

f_n' is h.w. vector of h.w. $(1^n)\,(1^{n-1}, -1)$

Lemma. If m odd, $B \times B'$ orbit of J open, dense

If m is even, $B \times B'$ orbits of J, J' are open and each one is dense in their respective irred. comp.

Pf. (idea): $(B \times B') \cdot J = \{u \mid f_i(u) \neq 0 \ i = 1 \dots n\}$

$$(B \times B') \cdot J' = \left\{ u \;\middle|\; \begin{matrix} f_i(u) \neq 0 \ \ i = 1 \dots n-1 \\ f_n'(u) \neq 0 \end{matrix} \right\} \quad \square$$

Lemma. Let (λ, λ') be h.w. in $\mathbb{C}[X]$. Then, $\lambda_i = \lambda_i'$ for $i = 1, \dots, n-1$. If m odd, $\lambda_n = \lambda_n'$.

If m even, $\lambda_n = |\lambda_n'|$.

Cor. (Orthogonal Cauchy identity) We have an isomorphism of $GL_n\mathbb{C} \times SO_m\mathbb{C}$-reps:

$$\mathbb{C}[X] \cong \bigoplus_{\lambda} S_\lambda \mathbb{C}^m \otimes S_{[\lambda]} \mathbb{C}^m$$

$\lambda_1 \geq \cdots \geq \lambda_n \geq 0$

Note: if $m = 2n$, $\lambda_n > 0$, $S_{[\lambda]}\mathbb{C}^m = S_{[\lambda]^+}\mathbb{C}^m \oplus S_{[\lambda]^-}\mathbb{C}^m$

Pf. If m is odd, proceed as before (X irred)

If m is even: X has 2 components X^+, X^-

X^+ containing J has h.w. $f_1 \ldots f_n$

$$\mathbb{C}[X^+] = \bigoplus_{\lambda} S_\lambda \mathbb{C}^m \otimes S_{[\lambda]^+} \mathbb{C}^m$$

X^- contains J' has h.w. $f_1, \ldots, f_{n-1}, f_n'$.

$$\mathbb{C}[X^-] \cong \bigoplus_{\lambda} S_\lambda \mathbb{C}^m \otimes S_{[\lambda]^-} \mathbb{C}^m$$

have surjective maps $\mathbb{C}[X] \to \mathbb{C}[X^+]$, $\mathbb{C}[X] \to \mathbb{C}[X^-]$

$\Rightarrow$ all h.w. we need exist on $\mathbb{C}[X]$. $\square$