Math 251C, Lecture 17

Note Title 5/6/2020

Homogeneous Spaces

Projective space U = vector space,

$\mathbb{P}(U)$ = set of 1-dim subspaces of U

If $U = \mathbb{C}^{n+1}$, let $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$.

$\mathbb{P}(U) = (U \setminus 0)/\mathbb{C}^*$ inherits quotient topology

$\rightsquigarrow$ coordinates on $\mathbb{P}^n$ via $[a_0 : \cdots : a_n]$

where $[a_0 : \cdots : a_n] = [\lambda a_0 : \cdots : \lambda a_n]$ for $\lambda \neq 0$, least one $a_i \neq 0$.

Topology on $\mathbb{P}^n$ is given by setting closed subsets to be $Z(I)$, $I \subset \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous ideal. $Z(I) \leftarrow$ projective varieties

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Prop. $\dim \mathbb{P}^n = n$, $\mathbb{P}^n$ is irreducible.

Pf. $GL_{n+1}\mathbb{C}$ acts on $\mathbb{P}^n$ transitively. $\Rightarrow$ irred

$$\mathrm{stab}(\langle e_1 \rangle) = \begin{pmatrix} * & * & * \cdots * \\ 0 & & \\ 0 & & * \\ \vdots & & \\ 0 & & \end{pmatrix} \leftarrow \dim = (n+1)^2 - n$$

$\Rightarrow \dim \mathbb{P}^n = \dim GL_{n+1} - \dim \mathrm{stab} = n$ $\square$

For each $i = 0, \ldots, n$, let $U_i := \{a \in \mathbb{P}^n \mid a_i \neq 0\}$
given point $a \in U_i$, can rescale a_i to be 1. The remaining coordinates gives identification $U_i \cong \mathbb{C}^n$.

$\mathbb{P}^n = U_0 \cup \cdots \cup U_n$ (in Euclidean topology, these give charts for $\mathbb{P}^n$ as complex manifold.)

$$\mathbb{P}^n = U_n \amalg \{a \mid a_n = 0\}$$
$$\cong \mathbb{C}^n \amalg \mathbb{P}^{n-1}$$
$$\cdots \cong \mathbb{C}^n \amalg \mathbb{C}^{n-1} \amalg \mathbb{C}^{n-2} \amalg \cdots \amalg \mathbb{C}^0$$

where for $i < n$, $\mathbb{C}^i = \{a \in \mathbb{P}^n \mid a_{i+1} = \cdots = a_n = 0, a_i \neq 0\}$.

Each $\mathbb{C}^i$ is an orbit under Borel subgroup of upper triangular matrices in $GL_{n+1}\mathbb{C}$. (Bruhat decomposition)

<u>Grassmannians</u>. $Gr(k, U)$ = set of k-dim subspaces of U.

$Gr(1, U) = \mathbb{P}(U)$. $Gr(k, U) = Gr(\dim U - k, U^*)$

$$W \longrightarrow \{f \in U^* \mid f(w) = 0 \ \forall w \in W\}$$

For $U = \mathbb{C}^n$, let $Gr(k, n) = Gr(k, \mathbb{C}^n)$.

Can realize $Gr(k, n)$ as a quotient space.

let $\mathbb{C}^{k \times n}$ = space of $k \times n$ matrices.

Given subspace $W \subset \mathbb{C}^n$, pick basis for W
$\rightsquigarrow$ get $k \times n$ matrix whose rows are this basis.
All of them are related by invertible row operations, i.e., an action of $GL_k\mathbb{C}$.

Let $(\mathbb{C}^{k\times n})^\circ$ = set of full rank matrices.

$$Gr(k,n) = (\mathbb{C}^{k\times n})^\circ / GL_k\mathbb{C}.$$

Another approach:

$W \subset \mathbb{C}^n$ k-dim $\Longrightarrow$ $\bigwedge^k W \subset \bigwedge^k \mathbb{C}^n$ 1-dim

Explicitly, if $w_1 \dots w_k$ basis for W, then $\bigwedge^k W$ is line spanned by $w_1 \wedge \dots \wedge w_k$. If we pick different basis, it is of the form $g(w_1) \dots g(w_k)$ for some $g \in GL(W)$

then $g(w_1) \wedge \dots \wedge g(w_k) = \det(g)\, w_1 \wedge \dots \wedge w_k$.

$$\begin{aligned} Gr(k,n) &\longrightarrow \mathbb{P}\left(\textstyle\bigwedge^k \mathbb{C}^n\right) \quad \text{injective.} \\ W &\longmapsto \textstyle\bigwedge^k W. \end{aligned}$$

Plücker embedding

More explicitly, we have basis of $\bigwedge^k \mathbb{C}^n$ indexed by k-element subsets of $\{1, \dots n\}$. $\longrightarrow$ projective coordinates for $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$.

Given subspace $W \subset \mathbb{C}^n$, realize it as row space of a $k \times n$ matrix. For every k-element subset $I \subset \{1, \dots, n\}$

let $f(W)_I = \det \begin{pmatrix} \text{submatrix whose columns} \\ \text{indexed by } I \end{pmatrix}$

If we pick different matrix, all of the $f(W)_I$ get changed by $\det(g)$ where g row operation.

$$\begin{aligned} Gr(k,n) &\longrightarrow \mathbb{P}^{\binom{n}{k}-1} \\ W &\longrightarrow (f(W)_I). \end{aligned}$$

Plücker embedding in coordinates

Plücker coordinates of W

<u>Prop</u>. $Gr(k,n)$ is a projective subvariety of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$.

<u>Pf</u>. Image of $Gr(k,n)$ is set of lines spanned by totally decomposable tensors $w_1 \wedge \dots \wedge w_k$.

Define $\varphi : \bigwedge^k \mathbb{C}^n \longrightarrow \mathrm{Hom}(\mathbb{C}^n, \bigwedge^{k+1} \mathbb{C}^n)$

$$\alpha \longrightarrow \varphi(\alpha)(u) = \alpha \wedge u$$

Note: $\cdot\ \alpha \wedge u = 0 \iff \alpha = \beta \wedge u$ for some $\beta \in \Lambda^{k-1}\mathbb{C}^n$ (exercise)

$\Rightarrow$ If $\alpha = w_1 \wedge \dots \wedge w_k$, then $\ker \varphi(\alpha) = \operatorname{span}(w_1, \dots w_k)$.

$\Rightarrow \operatorname{rk} \varphi(\alpha) = n-k$

Conversely, if α is not totally decomposable, then $\dim \ker \varphi(\alpha) < k$. If $w_1, \dots w_k \in \ker \varphi(\alpha)$ linearly independent, $\Rightarrow \alpha = w_1 \wedge \dots \wedge w_k \cdot$ Scalar

$\Rightarrow \alpha$ totally decomposable $\iff \operatorname{rk} \varphi(\alpha) \le n-k$

$\iff$ all submatrices of $\varphi(\alpha)$ of size $n-k+1$ have det 0

these dets are homog. poly

$\Rightarrow Gr(k,n) = Z(\text{dets})$ $\square$

Example: $Gr(1,n) = \mathbb{P}^{n-1} \cong Gr(n-1,n)$

$k=2,\ n \ge 4$ is new. $\Lambda^2 \mathbb{C}^n$ = space of skew-sym $n \times n$ matrices.

totally decomposable $\iff u \wedge v$ = rank 2 matrix.

$Gr(2,n)$ = set of rank 2 skew-sym matrices / scaling

$= Z(4 \times 4$ Pfaffians of principal submatrices$)$

If $k=2, n=4$, $Gr(2,4) =$ rk 2 matrices in $\overset{2}{\wedge}\mathbb{C}^4$

There is 1 Pfaffian whose equation is

$$X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} = 0$$

<u>Prop</u>. $\dim Gr(k,n) = k(n-k)$, $Gr(k,n)$ is irred.

<u>Pf</u>. $GL_n\mathbb{C}$ acts transitively on $Gr(k,n) \Rightarrow$ irred.

$$\operatorname{stab}(\langle e_1, \dots, e_k \rangle) = \begin{array}{cc} \begin{array}{cc} k & n-k \end{array} & \\ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) & \begin{array}{c} k \\ n-k \end{array} \end{array}$$

$$\dim \operatorname{stab}(\langle e_1 \dots e_k \rangle) = n^2 - k(n-k)$$

$$\dim Gr(k,n) = \dim GL_n - \dim \operatorname{stab} = k(n-k). \quad \square$$