Math 251C, Lecture 19

Note Title 5/11/2020

Consider $\mathbb{P}(S_\lambda \mathbb{C}^n)$, define

highest weight orbit

$$X_\lambda = \left\{ [x] \in \mathbb{P}(S_\lambda \mathbb{C}^n) \;\middle|\; \begin{array}{l} x \text{ is a h.w. vector for} \\ \text{some choice of Borel subgp} \end{array} \right\}$$

<u>Lemma</u>. X_λ is closed under the action of $GL_n\mathbb{C}$ and is a single orbit.

<u>Pf</u>. Let v = h.w. vector for a Borel $B \subseteq GL_n\mathbb{C}$.
$\Rightarrow$ gv is a h.w. vector for gBg^{-1}, so X_λ is closed under $GL_n\mathbb{C}$.

Let w be another h.w. vector wrt Borel B'.
$\exists g \in GL_n\mathbb{C}$ s.t. $B' = gBg^{-1} \Rightarrow gbg^{-1}[w] = [w]$ $\forall b \in B$

Equivalently, $bg^{-1}[w] = g^{-1}[w]$ $\forall b \in B$

i.e., $g^{-1}w$ is a h.w. vector for B.

By uniqueness of h.w. vectors, $[g^{-1}w] = [v]$. $\square$

Consider $\lambda = (1^d)$. $S_\lambda \mathbb{C}^n = \bigwedge^d \mathbb{C}^n$. wrt to standard Borel, $e_1 \wedge \cdots \wedge e_d$ is a h.w. vector.

$$X_{1^d} = \{ [w_1 \wedge \cdots \wedge w_d] \mid w_i \text{ lin. ind.} \} = Gr(d, n).$$

The stabilizer of $[e_1 \wedge \cdots \wedge e_d]$ is $P_d = \begin{array}{cc} \begin{array}{cc} d & n-d \end{array} & \\ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) & \begin{array}{c} d \\ n-d \end{array} \end{array}$

General case: recall $S_\lambda \mathbb{C}^n$ is a quotient of $\bigwedge^{\mu_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\mu_r} \mathbb{C}^n$ where $\mu = \lambda^T$.

Taking image of tensor product of h.w. vectors is a h.w. vector for $S_\lambda \mathbb{C}^n$.

$\Rightarrow$ Stab of the line spanned by this h.w. vector is

$$P_{\mu_1} \cap P_{\mu_2} \cap \cdots \cap P_{\mu_r} = \begin{array}{cc} & \begin{array}{cccc} \mu_r & \mu_r - \mu_{r-1} & \cdots & n-\mu_1 \end{array} \\ \begin{array}{c} \mu_r \\ \mu_r - \mu_{r-1} \\ \vdots \\ n-\mu_1 \end{array} & \left(\begin{array}{c|c|c|c} * & * & * & * \\ \hline 0 & * & * & * \\ \hline 0 & 0 & \cdots & * \\ \hline 0 & 0 & 0 & * \end{array}\right) \end{array}$$

This is the stabilizer of the standard partial flag of dimensions $\mu_1, \mu_2, \ldots, \mu_r$ $\Longrightarrow X_\lambda \cong Fl(\underbrace{\mu_1, \ldots, \mu_r}_{\text{sort}}; n)$

Prop. $X_\lambda \cong Fl(\underline{d};n)$ where $\underline{d}$ is obtained from λ^T by removing repetitions and sorting in increasing order.

Ex. $\lambda=(1^d)$, $\lambda^T=(d) \Rightarrow X_\lambda = Gr(d,n)$

$\lambda=(2,1)$, $\lambda^T=(2,1) \Rightarrow X_\lambda \cong Fl(1,2;n)$

$\lambda=(2,2,1)$ $\lambda^T=(3,2) \Rightarrow X_\lambda \cong Fl(2,3;n)$

$\lambda=(2^d)$ $\lambda^T=(d,d) \Rightarrow X_\lambda = Gr(d,n)$

Homogeneous bundles. $G=GL_n\mathbb{C}$, $P=P_{\underline{d}}$, $X=G/P$

Def. $\pi: E \to X$ vector bundle is **homogeneous** if G acts on E (algebraically) s.t. π is G-equivariant.

$\pi(ge) = g\pi(e)$ $\forall g \in G$ $\forall e \in E$

$\Rightarrow$ $\forall x \in X$, $E_x = \pi^{-1}(x)$ is a rat'l rep. of $\mathrm{stab}(x)$

Fix $x \in X$.

Thm. "E is completely encoded by rep. E_x of $\mathrm{stab}(x)$"

Formally: The category of homog. bundles is equivalent to the category of rat'l reps of $\mathrm{stab}(x)$.

Given a rat'l rep V of P = stab(std. flag)

$$\mathrm{Ind}_P^G(V) := (G \times V)/\sim$$

where $(g, v) \sim (gp^{-1}, pv)$ $\forall p \in P, g \in G, v \in V$

- Define $\pi : \mathrm{Ind}_P^G(V) \longrightarrow G/P$ by

$$\pi(g, v) = gP$$

- well-defined: $\pi(gp^{-1}, pv) = gp^{-1}P = gP$.
- G acts on $\mathrm{Ind}_P^G(V)$ via $g' \cdot (g, v) = (g'g, v)$.

  π is G-eqvt.
- $\pi^{-1}(gP) = \{(gp, v) \mid p \in P, v \in V\}/\sim$

$$= \{(g, v) \mid v \in V\} \cong V$$

$\Longrightarrow$ $\mathrm{Ind}_P^G(V)$ is a homog. bundle.

s.t. $\mathrm{Ind}_P^G(V)\big|_P = V$ as rep.

We will omit check that $\mathrm{Ind}_P^G(E_x) = E$ $\forall$ homog. E

$\pi: E \to X$ vector bundle. Let $H^0(X;E)$ = set of sections $s: X \to E$, i.e., $\pi s = id_X$.

complex vector space. (in fact, finite dim. since X proj.)

If E homog. then $H^0(X;E)$ is a rat'l rep of G.

<u>Borel-Weil Theorem</u>. $T \subset B \subset G$

max torus Borel

$T \cong B/[B,B]$, so we have surjection $B \twoheadrightarrow T$.

Given $\lambda \in \mathbb{Z}^n$, we get 1-dim rep $\mathbb{C}_\lambda$ of T by

$t \cdot z = \lambda(t) z \quad \forall t \in T, \ z \in \mathbb{C}_\lambda$.

Via $B \twoheadrightarrow T$, this gives 1-dim rep of B (call it $\mathbb{C}_\lambda$)

$\mathcal{L}_\lambda := \mathrm{Ind}_B^G \mathbb{C}_\lambda$. line bundle on G/B (i.e., rank 1 vector bundle).

<u>Thm</u> (Borel-Weil). If λ dominant, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then $H^0(G/B; \mathcal{L}_\lambda)^* \cong S_\lambda \mathbb{C}^n$.

<u>Pf</u>. Let L = subgroup of $GL_n\mathbb{C}$ of lower-triangular matrices w/ 1's on the diagonal. We have natural correspondence:

h.w. vectors in $H^0(G/B; \mathcal{L}_\lambda)^*$ $\longleftrightarrow$ L-fixed vectors in $H^0(G/B; \mathcal{L}_\lambda)$

WTS: $\exists$ unique up to scalar, L-fixed vector in H^0.

Recall: set of matrices of the form lb, $l \in L$, $b \in B$ is open and dense in space of matrices.

$\Longrightarrow \{lB \mid l \in L\} \subset G/B$ is dense

If $s \in H^0(G/B; \mathcal{L}_\lambda)$ is L-fixed, then

$$s(B) = (l \cdot s)(B) = l(s(\underbrace{l^{-1}B}_{\text{dense ranging over } l \in L})) \quad \forall l \in L.$$

$\Longrightarrow$ s is determined by its value on $B \in G/B$.

Since $s(B) \in \mathbb{C}_\lambda$ $\Longrightarrow$ L-fixed vectors unique up to scalar.

$\Longrightarrow$ $H^0(G/B; \mathcal{L}_\lambda)^*$ is irreducible.