Math 251C, Lecture 21

Note Title 5/15/2020

Lemma. Given $(n-1)$-dim'l isotropic subspace W of $\mathbb{C}^{2n}$, $\exists$ exactly 2 n-dim'l isotropic subspaces containing W.

Let $\underline{d}$ be a subset of $\{1, 2, \ldots, n-2, n, n'\}$

$OF\ell(\underline{d}; 2n) = \{ W_i \mid i \in \underline{d}$ s.t. ①,②,③ hold$\}$

① $\dim W_i = i$ for $i \leq n-2$; $\dim W_n = n$, $\dim W_{n'} = n$ — whenever these appear in $\underline{d}$

$\dim(W_n \cap \operatorname{span}\{e_1, \ldots, e_n\})$ even

$\dim(W_{n'} \cap \operatorname{span}\{e_1, \ldots, e_n\})$ odd

② If $i < j$ then $W_i < W_j$ ($i < n'$ for all $i \leq n-2$)

③ If $n, n' \in \underline{d}$, then $\dim(W_n \cap W_{n'}) = n-1$.

For any such $\underline{d}$, $OF\ell(\underline{d}; 2n)$ is homogeneous space for $SO(2n)$.

Rmk. If both $n, n' \in \underline{d}$, can replace them by $n-1$ and $W_{n-1} = W_n \cap W_{n'}$.

Prop. $OF\ell(\underline{d};m)$ is projective variety.

Rmk. If E v.bundle on space X, it is <u>orthogonal</u> if have linear map $\mathrm{Sym}^2 E \longrightarrow \mathcal{L}$, some line bundle $\mathcal{L}$ on X s.t. on each fiber this gives orthogonal form

Can define relative $OF\ell(\underline{d};E)$, has map to X whose fibers are $OF\ell(\underline{d};E_x)$.

If $\underline{d}=(1)$, then $OF\ell(\underline{d};m)$ is solution of

$$\sum_{i=1}^{m-n} x_i x_{m+1-i} = 0 \text{ in } \mathbb{P}^{m-1},$$ so $\dim = m-2$.

Prop. $\dim OF\ell(m) = n(m-n-1)$.

Pf. Let R be tautological subbundle on $OF\ell(1;m)$. $R^\perp/R$ orthogonal bundle of rank $m-2$, and

$OF\ell(m) = OF\ell(R^\perp/R)$, fibers isomorphic to $OF\ell(m-2)$. $\Longrightarrow$

$\dim OF\ell(m) = (m-2) + \dim OF\ell(m-2)$.

<u>Base cases</u>: $m=2$, $OF\ell(2) = 2$ points, 0-dim ✓

$m=3$, $OF\ell(3)$ = curve in $\mathbb{P}^2$ 1-dim ✓. $\square$

Rmks. ① OFl, IFl can be realized as h.w. orbits in irred. reps of SO, Sp.
② Borel-Weil is valid

Kempf-Weyman collapsing

U = vector space, X = projective variety
trivial v.bundle $U \times X$.
$S \subset U \times X$ subbundle.
$S \longrightarrow U$ via projection onto first factor
(Since X is projective, image Y is closed)
Y is the collapsing of S.

[In good cases, can compute things about Y, such as its coordinate ring, defining ideal, ... using calculations on X, S, related v. bundles]

If X is irreducible, then so is S, and so is Y.

Example (determinantal varieties)

E, F vector spaces, $U = \mathrm{Hom}(E,F)$.

Assume $\dim E \leq \dim F$. Pick $r < \dim E$.

Define $Y = \{ \varphi : E \to F \mid \mathrm{rk}\,\varphi \leq r \} \subseteq U$ affine variety.

Y is not linear, can try to "linearize"

Fix an r-dim'l subspace $W \subset F$. Then the set of linear maps $E \to W$ is linear space, contained in Y

If we vary choice of W, get all of Y.

(Alternatively. $GL(F) \cdot \mathrm{Hom}(E,W) = Y$.)

X = choices of W = $Gr(r, F)$

$S = \mathrm{Hom}(E \times X, R)$ $\quad R$ taut. subbundle.

Note that $S \subset U \times X$: given $\varphi : E \times X \to R$, compose w/ inclusion $R \subset U \times X$

What is image of S in U? $= \{ \varphi : E \to F \mid \exists\, W \in Gr(r,F),\ \text{image } \varphi \subseteq W \}$

$= \{ \varphi \mid \mathrm{rk}\,\varphi \leq r \} = Y$.

$S = \{(f, W) \mid W \in Gr(r, F),\ f : E \to W\}$

$\pi : S \longrightarrow U$ is $\pi(f, W) = f$.

what is $\pi^{-1}(f)$? $= \{(f, W) \mid \begin{array}{l} \dim W = r \\ W \supseteq \operatorname{im} f \end{array}\}$

If $\operatorname{rk} f = r$, then $\pi^{-1}(f)$ is 1 point.

$\Longrightarrow$ π is birational isomorphism

(i.e., isom. over an open set)

[Since X is homogeneous space, it is automatically smooth $\Rightarrow$ S is also smooth $\Rightarrow$ π is a desingularization of Y]

Analogous example for (skew-)symmetric matrices of rank $\leq r$.

Example (isotropic maps) E = vector space, m-dim'l

V = symplectic space of dim $2n$. $U = \operatorname{Hom}(E, V)$.

$Y = \{\varphi : E \to V \mid \operatorname{im} \varphi \text{ is isotropic}\} \subset U$.

$X = IFl(\min(m, n); V)$, $S = \operatorname{Hom}(E, R)$

image of S in U is Y, desingularization.

__Example__ (Nilpotent cone)

E vector space of dim n. $U = \mathrm{End}(E) = \mathrm{Hom}(E,E)$

$x \in U$ __nilpotent__ if some power of x is 0

$$\iff x^n = 0.$$

$Y = \{x \mid x \text{ nilpotent}\}$ affine variety:

entries of x^n are polynomials in entries of x

$$Y = Z(\qquad).$$

Alternatively: x nilpotent $\iff$ all eigenvalues of x are 0

$$\iff \mathrm{char}(x) = t^n.$$

$$\mathrm{char}(x) = \det(tI_n - x) = \sum_{i=0}^{n} a_i(x)\, t^i$$

where a_i are polynomials in entries of x.

(e.g. $a_0 = (-1)^n \det(x)$, $a_{n-1} = -\mathrm{trace}(x)$)

$$Y = Z(a_0, a_1 \dots a_{n-1}).$$