Math 251C, Lecture 22

Note Title 5/18/2020

Example (Nilpotent cone)

E vector space of dim n. $U = \mathrm{End}(E) = \mathrm{Hom}(E,E)$

$Y = \{x \mid x \text{ nilpotent}\}$ affine variety

Pick a basis for E, x strictly upper triangular $\Rightarrow x$ nilpotent

$\{$strictly upper-triangular matrices$\}$ is linear subspace of Y.

If x is nilpotent, then we get flag

$\mathrm{image}\, x \supset \mathrm{image}\, x^2 \supset \cdots \supset \mathrm{image}\, x^{n-1}$

If $\mathrm{image}\, x^i \neq 0$, then it strictly contains $\mathrm{image}\, x^{i+1}$

$\rightsquigarrow$ partial flag if we truncate where $x^i = 0$.

Can refine this to a complete flag, wrt this flag, x is strictly upper-triangular.

$\Rightarrow Y = GL(E) \cdot \left\{ \begin{array}{l} \text{strictly upper-triangular} \\ \text{matrices} \end{array} \right\}$

$X = F\ell(E)$

$$S = \{(x, F_\cdot) \mid x(F_i) \subseteq F_{i-1} \ \forall i\} \subseteq U \times X$$

The fiber of S over $F_\cdot$ is all x which are strictly upper-triangular wrt $F_\cdot$ $\Rightarrow$ S is v. bundle.

image of $\pi: S \to U$ is Y.

- Y is irreducible
- π is birational isomorphism ($\pi^{-1}(x)$ is one point if Jordan normal form of x has 1 block)

$\Rightarrow \dim Y = \dim X + \operatorname{rk} S = \binom{n}{2} + \binom{n}{2} = n^2 - n$

Note: $\dim U = n^2$, $Y = Z$(n coeffs of char. poly)

can show coeffs of char. poly generate full ideal of Y.

Rmk. Can stratify nilpotent matrices by the sizes and # blocks in Jordan normal form.

Same framework, but use partial flag varieties instead

Example (Binary forms) $U = \mathrm{Sym}^d \mathbb{C}^2$

$=$ homog. deg d polys in x, y.

Every polynomial factors into product of linear forms.

Fix $2 \le p \le d$. $Y = \{\varphi \mid \varphi = \ell^p f,\ \ell \text{ linear}\} \subseteq U$

$\{x^p f \mid \deg f = d-p\}$ linear subspace

$GL_2(\mathbb{C}) \cdot \{x^p f\} = Y$

$X = \mathbb{P}^1 = \mathbb{P}(U)$ (choice of ℓ)

$S = \{([\ell], \ell^p f)\} \subseteq U \times X$

image of $\pi : S \to U$ is Y

Y is irreducible, π is birational isomorphism

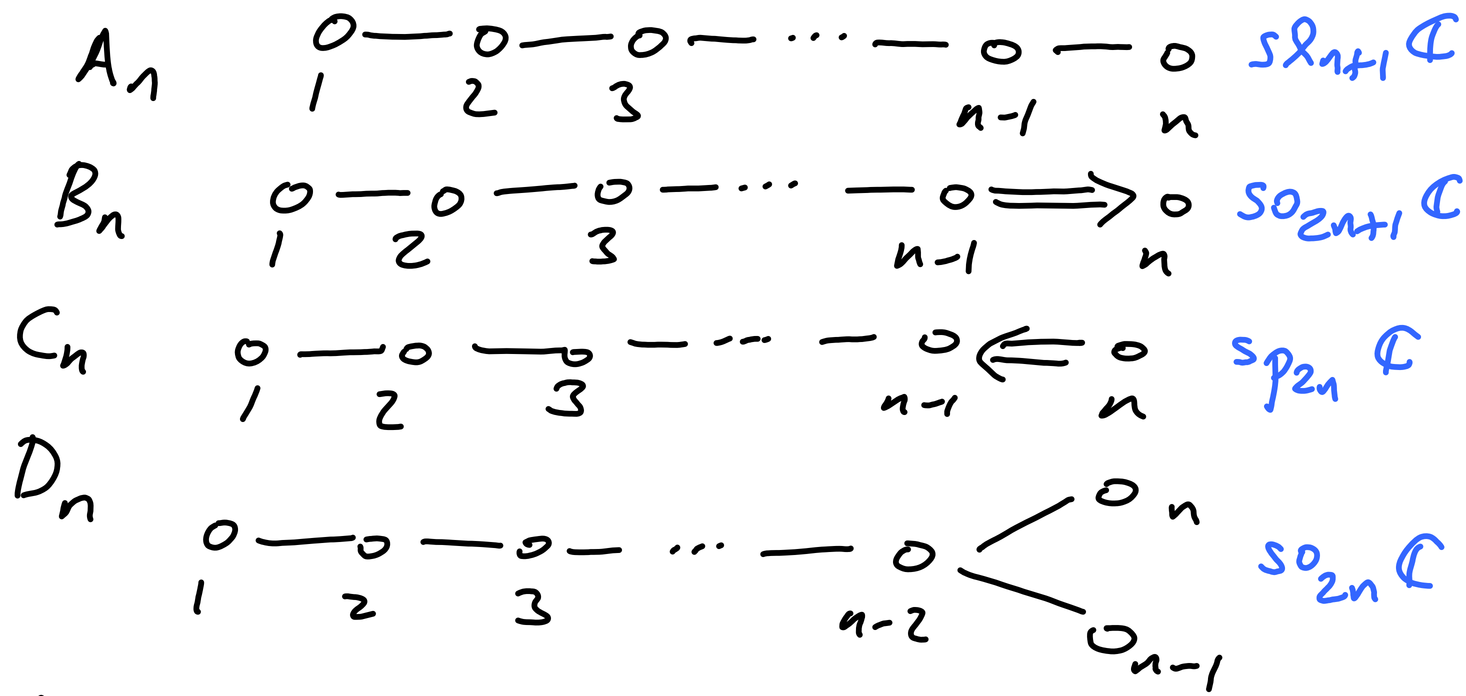
Dynkin diagrams

Dynkin diagrams $\longleftrightarrow$ simple Lie algebras / $\mathbb{C}$

4 infinite families

+ 5 exceptional

$\longleftrightarrow$ simple, simply-connected Lie groups / $\mathbb{C}$.



A_n: $SL_{n+1}\mathbb{C}$

B_n: $Spin_{2n+1}\mathbb{C}$ (double cover of $SO_{2n+1}\mathbb{C}$)

C_n: $Sp_{2n}\mathbb{C}$

D_n: $Spin_{2n}\mathbb{C}$ (double cover of $SO_{2n}\mathbb{C}$)

n = dim of a maximal torus

Each node i corresponds to weight ω_i
= highest weights of "fundamental representations"

If f function from nodes to $\mathbb{Z}_{\geq 0}$, get dominant weight

$$\sum_i f(i)\,\omega_i = \text{highest weight of an irred. rep.}$$

- A_n, $\omega_i = (1^i, 0^{n-i}) \leftrightarrow \bigwedge^i \mathbb{C}^{n+1}$

(note for $SL_{n+1}\mathbb{C}$, $\bigwedge^{n+1} \mathbb{C}^{n+1}$ is trivial)

$\sum f(i)\omega_i \longleftrightarrow \lambda$ where $\lambda_j = f(1) + \dots + f(j)$

- B_n, for $i < n$, $\omega_i = (1^i, 0^{n-i}) \leftrightarrow \bigwedge^i(\mathbb{C}^{2n+1})$.

$\omega_n = \frac{1}{2}(1^n) \leftrightarrow$ "Spin rep"

$\left\{\begin{array}{l}\text{Irred reps}\\ \text{of } SO_{2n+1}\mathbb{C}\end{array}\right\} \longleftrightarrow \{f \text{ s.t. } f(n) \text{ is even}\}$

$(2\omega_n \longleftrightarrow \bigwedge^n \mathbb{C}^{2n+1})$

$\left\{\begin{array}{l}\text{Irred reps of}\\ Spin_{2n+1}\mathbb{C} \text{ that}\\ \text{don't factor through } SO_{2n+1}\mathbb{C}\end{array}\right\} \longleftrightarrow \{f \mid f(n) \text{ is odd}\}$

- C_n, $\omega_i = (1^i, 0^{n-i}) \leftrightarrow \bigwedge_0^i \mathbb{C}^{2n} = \mathrm{coker}(\bigwedge^{i-2}\mathbb{C}^{2n} \to \bigwedge^i \mathbb{C}^{2n})$

- D_n. for $i \leq n-2$, $\omega_i = (1^i, 0^{n-i}) \leftrightarrow \bigwedge^i \mathbb{C}^{2n}$

$\omega_{n-1} = \frac{1}{2}(1^n)$, $\omega_n = \frac{1}{2}(1^{n-1}, -1)$.

$\left\{\begin{array}{l}\text{Irred reps of}\\ SO_{2n}\mathbb{C}\end{array}\right\} \longleftrightarrow \{f \mid f(n-1) + f(n) \text{ is even}\}$

$\left.\begin{array}{l}\omega_{n-1}\\ \omega_n\end{array}\right. \longleftrightarrow$ half-spin reps

There is "standard maximal torus"

Each node $\Longleftrightarrow$ pair of "root subgroups" (positive/negative)

$$SL_{n+1}\mathbb{C}: \quad \underset{i}{\circ} \longleftrightarrow \left\{ \begin{array}{cc} & \scriptstyle i+1 \\ \scriptstyle i & \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & t & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array} \middle| \; t \in \mathbb{C} \right\}$$

positive root subgp

negative = transpose(positive)

- maximal torus + positive root subgps generate a Borel subgp "standard"
- maximal torus + negative root subgps generate a Borel subgp "opposite"
- nonempty subset S of nodes $\longleftrightarrow$ Standard Borel + negative root subgps indexed by S generates a parabolic subgp P_S. "standard"

$$\{\text{homog. spaces}\} \longleftrightarrow \left\{\begin{array}{c}\text{conj. classes} \\ \text{of parabolic} \\ \text{subgps}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\neq \emptyset \text{ subsets} \\ \text{of Dynkin} \\ \text{diagram}\end{array}\right\}$$

Each node $\leftrightarrow$ certain kind of subspace.

A_n: $i \leftrightarrow i$-dim subspaces

B_n, C_n: $i \leftrightarrow i$-dim isotropic subspace

D_n: If $i \leq n-2$, $i \leftrightarrow i$-dim isotropic subspace

$n-1 \leftrightarrow$ odd n-dim isotropic subspace

$n \leftrightarrow$ even n-dim isotropic subspace