Math 251C, Clifford algebras

Note Title 5/27/2020

<u>Setup</u>: V orthogonal space of dim m,
ω orthogonal form.

$$T(V) = \bigoplus_{d\geq 0} V^{\otimes d}$$ w/ concatenation of tensors as product

(tensor algebra)

$C(V,\omega) = C(V)$ is the quotient of $T(V)$ by 2-sided ideal generated by relations

$$vw + wv = 2\omega(v,w) \quad v,w \in V^{\otimes 1} = V$$

(Clifford algebra)

<u>Note</u>. $\deg(vw+wv) = 2$, $\deg(2\omega(v,w)) = 0$, so not homogeneous relation for $\mathbb{Z}$-grading of $T(V)$, but is homogeneous for $\mathbb{Z}/2$-grading

$$\Rightarrow C(V) = C^{+}(V) \oplus C^{-}(V)$$

$C^{+}(V)$: even elements; $C^{-}(V)$: odd elements

$C(V)$ has the following "universal property":

If A is any associative algebra, then the data of a homomorphism $\varphi: C\ell(V) \to A$ is the same as a linear map $\varphi: V \to A$ s.t.

$$\varphi(v)\varphi(w) + \varphi(w)\varphi(v) = 2\omega(v,w). \quad \forall v, w \in V.$$

<u>Lemma</u>. Let $v_1, \ldots v_m$ be a basis of V.

For $I = (i_1, \ldots, i_k)$ w/ $1 \le i_1 < i_2 < \ldots < i_k \le m$,

define $v_I = v_{i_1} v_{i_2} \cdots v_{i_k} \in C(V)$. $[v_\emptyset = 1]$

Then $\{v_I\}$ is a basis for $C(V)$.

Hence $\dim C(V) = 2^m$.

<u>Pf</u>. Taking all products of v_i's gives basis for $T(V)$, hence their images in $C(V)$ span.

The Clifford relation $\Rightarrow$ if $i > j$, then

$v_i v_j = -v_j v_i + 2\omega(v_i, v_j) \Rightarrow$ can rewrite any product in order.

Furthermore, $v_i^2 + v_i^2 = 2\omega(v_i, v_i) \Rightarrow$

$$v_i^2 = \omega(v_i, v_i).$$

$\Longrightarrow \{v_I \mid I = 1 \le i_1 < \cdots < i_k \le m\}$ span $C(V)$.

For each d, define $C^{\le d}(V) =$ span of all $w_1 \cdots w_k$ where $w_i \in V$ and $k \le d$.

($=$ image of $V^{\otimes 0} \oplus V^{\otimes 1} \oplus \cdots \oplus V^{\otimes d}$ in $C(V)$)

$$\Longrightarrow \quad C(V) \cong \bigoplus_{d \ge 0} C^{\le d}(V) / C^{\le d-1}(V)$$

as vector spaces

satisfies the relations for exterior algebra $\Lambda(V)$

In $\Lambda(V)$, v_I linearly independent, so they were linearly ind. to begin with. $\square$

Consider $m = 2n$, even.

Write $V = W \oplus W'$ where W, W' are both isotropic of dimension n.

(if $\{e_1, \dots, e_{2n}\}$ is hyperbolic basis, can take

$W = \text{span}(e_1, \dots, e_n)$, $W' = \text{span}(e_{n+1}, \dots, e_{2n})$.)

<u>Note:</u> $W' \cong W^*$ via ω.

$$\Lambda^{\cdot}(W) = \bigoplus_{d \geq 0} \Lambda^{d}(W)$$ exterior algebra (product given by concatenating tensors)

<u>Thm</u> If $\dim V = 2n$, then $C(V) \cong \text{End}(\Lambda^{\cdot} W)$

i.e., $C(V) \cong$ algebra of matrices of size 2^n.

and $C(V)$ is simple.

<u>Pf</u>. First, construct $\varphi : C(V) \to \text{End}(\Lambda^{\cdot} W)$

using universal property of $C(V)$.

For $w \in W$, let $\varphi(w) \in End(\wedge W)$ be left multiplication by w, i.e.,

$$\varphi(w)(\alpha) = w \wedge \alpha.$$

For $w' \in W'$, let $\varphi(w') \in End(\wedge W)$ be

remove w_i

$$\varphi(w')(w_1 \wedge \cdots \wedge w_d) = 2\sum_{i=1}^{d} (-1)^{i-1} \omega(w', w_i)(w_1 \wedge \cdots \hat{w_i} \cdots \wedge w_d)$$

For $v = w + w'$, $w \in W$, $w' \in W'$, $\varphi(v) = \varphi(w) + \varphi(w')$.

φ is linear ✓

Need to check Clifford relations hold.

$$\varphi(v)\varphi(v') + \varphi(v')\varphi(v) = 2\omega(v, v').$$

Suffices to check when $v, v' \in W \cup W'$

① If $v, v' \in W$, then $\varphi(v)\varphi(v')(\alpha) = v \wedge v' \wedge \alpha$

$\varphi(v')\varphi(v)(\alpha) = v' \wedge v \wedge \alpha = -v \wedge v' \wedge \alpha$

$\Rightarrow \varphi(v)\varphi(v') + \varphi(v')\varphi(v) = 0 = 2\omega(v, v').$

② If $v, v' \in W'$, $\varphi(v)\varphi(v') + \varphi(v')\varphi(v) = 0$

③ If $v \in W$, $v' \in W'$, then

$$\varphi(v)\varphi(v')(w_1 \wedge \cdots \wedge w_d)$$

$$= \varphi(v)\sum_{i=1}^{d}(-1)^{i-1}2\omega(v', w_i)(w_1 \wedge \cdots \hat{w_i} \cdots \wedge w_d)$$

$$= \sum_{i=1}^{d}(-1)^{i-1}2\omega(v', w_i)(v \wedge w_1 \wedge \cdots \hat{w_i} \cdots \wedge w_d)$$

$$\varphi(v')\varphi(v)(w_1 \wedge \cdots \wedge w_d) = v \wedge w_1 \wedge \cdots \wedge w_d$$

$$= 2\omega(v', v)\, w_1 \wedge \cdots \wedge w_d + \sum_{i=1}^{d}(-1)^{i}2\omega(v', w_i)(v \wedge w_1 \wedge \cdots \hat{w_i} \cdots \wedge w_d)$$

$$\big(\varphi(v)\varphi(v') + \varphi(v')\varphi(v)\big)(w_1 \wedge \cdots \wedge w_d)$$

$$= 2\omega(v', v)(w_1 \wedge \cdots \wedge w_d)$$

$\Rightarrow$ $\varphi(v)\varphi(v') + \varphi(v')\varphi(v) = 2\omega(v', v)$.

$\Rightarrow$ homomorphism $\varphi : C(V) \to \mathrm{End}(\wedge W)$.

Claim. φ is injective.

Suppose $a = \sum_I \alpha_I e_I \in \ker \varphi$.

$I \leftarrow$ subsets of $\{1, \dots, 2n\}$

Define $s(I) = \#(I \cap \{n+1, \dots, 2n\})$

We will show by induction on $s(I)$ that $\alpha_I = 0$.

Base case: $s(I) = 0 \Rightarrow I \subseteq \{1, \dots, n\}$

$$\varphi(e_I)(1) = 0 \text{ if } s(I) > 0$$

$$\Rightarrow 0 = \varphi(a)(1) = \sum_I \alpha_I e_I$$

$I \leftarrow$ subsets of $\{1, \dots, n\}$

$\{e_I \mid I \subseteq \{1, \dots, n\}\}$ is a basis for $\Lambda(W)$

$\Rightarrow \alpha_I = 0 \quad \forall I$ s.t. $s(I) = 0$.

Induction step: assume $\alpha_I = 0 \ \forall I$ s.t. $s(I) \le d$.

$$\varphi(e_I)(e_{i_1} \wedge \dots \wedge e_{i_{d+1}}) = 0 \text{ if } s(I) > d+1$$

or if $s(I) = d+1$ and

$I \cap \{n+1, \dots, 2n\} \neq \{2n+1-i_1, \dots, 2n+1-i_{d+1}\}$

$$\varphi(e_{\{2n+1-i_1, \dots, 2n+1-i_{d+1}\}})(e_{i_1} \wedge \dots \wedge e_{i_{d+1}}) = \underset{\neq 0}{c_{i_1 \dots i_{d+1}}}$$

$\Longrightarrow \varphi(a)(e_{i_1} \wedge \cdots \wedge e_{i_{d+1}})$

$$= c_{i_1,\ldots,i_{d+1}} \sum_{I} \alpha_{I \cup \{2n+1-i_1,\ldots,2n+1-i_{d+1}\}} e_I$$

I ← subsets of $\{1,\ldots,n\}$

$\Longrightarrow$ α's appearing are 0. (since e_I are basis for $\wedge W$)

Now vary over all choices of $\{i_1,\ldots,i_{d+1}\}$ to finish induction.

$\Longrightarrow \varphi$ is injective.

Since $\dim C(V) = (2^n)^2 = \dim \operatorname{End}(\wedge W)$,

φ is an isomorphism. $\square$

Define $\bigwedge^{even} W = \bigoplus_{d \geq 0} \bigwedge^{2d} W$, $\bigwedge^{odd} W = \bigoplus_{d \geq 0} \bigwedge^{2d+1} W$.

<u>Cor.</u> $C^+(V) \cong \operatorname{End}(\bigwedge^{even} W) \times \operatorname{End}(\bigwedge^{odd} W)$.

So $C^+(V)$ is semisimple.

Defn $\overset{\text{even}}{\Lambda} W = \bigoplus_{d \geq 0} \overset{2d}{\Lambda} W$, $\overset{\text{odd}}{\Lambda} W = \bigoplus_{d \geq 0} \overset{2d+1}{\Lambda} W$.

Cor. $C^+(V) \cong \mathrm{End}(\overset{\text{even}}{\Lambda} W) \times \mathrm{End}(\overset{\text{odd}}{\Lambda} W)$.

So $C^+(V)$ is semisimple.

Pf. The action of $C^+(V)$ on $\overset{\cdot}{\Lambda} W$ preserves both $\overset{\text{even}}{\Lambda} W$ & $\overset{\text{odd}}{\Lambda} W$, so get

$$\varphi : C^+(V) \longrightarrow \mathrm{End}(\overset{\text{even}}{\Lambda} W) \times \mathrm{End}(\overset{\text{odd}}{\Lambda} W).$$

Since φ is injective on $C(V)$, it is also injective on $C^+(V)$.

$\dim C^+(V) = \frac{1}{2} \dim C(V) = 2^{2n-1}$

$\dim \overset{\text{even}}{\Lambda} W = 2^{n-1}$, $\dim \mathrm{End}(\overset{\text{even}}{\Lambda} W) \times \mathrm{End}(\overset{\text{odd}}{\Lambda} W)$

$\dim \overset{\text{odd}}{\Lambda} W = 2^{n-1}$ $\qquad = (2^{n-1})^2 + (2^{n-1})^2 = 2^{2n-1}$.

$\implies \dim C^+(V) = \dim(\mathrm{End}(\overset{\text{even}}{\Lambda} W) \times \mathrm{End}(\overset{\text{odd}}{\Lambda} W))$

$\implies \varphi$ is an isom. $\square$

Consider $\dim V = 2n+1$ odd.

Write $V = W \oplus W' \oplus L$, W, W' = n-dim isotropic spaces. $L = (W \oplus W')^\perp$

(if $\{e_1, \ldots, e_{2n+1}\}$ hyperbolic basis, then
$W = \text{span}(e_1, \ldots e_n)$, $L = \text{span}(e_{n+1})$
$W' = \text{span}(e_{n+2}, \ldots e_{2n+1})$).

<u>Thm</u>. $\dim V$ odd $\Rightarrow C(V) \cong \text{End}(\Lambda W) \times \text{End}(\Lambda W')$.

Hence $C(V) \cong$ product of 2 matrix algebras of size 2^n each,

$C(V)$ semisimple.

Furthermore $C^+(V) \cong \text{End}(\Lambda W) \cong \text{End}(\Lambda W')$.

<u>Pf</u>. Define $\varphi: C(V) \longrightarrow \text{End}(\Lambda W)$ using universal property. If $v \in W \oplus W'$, $\varphi(v)$ as before.

If $l \in L$ and $\omega(l, l) = 1$, define (for $\lambda \in \mathbb{C}$)

$$\varphi(\lambda l)(w_1 \wedge \cdots \wedge w_d) = \lambda (-1)^d w_1 \wedge \cdots \wedge w_d.$$

Clifford relations:

① $\varphi(\lambda l)\varphi(\mu l)(w_1\wedge\cdots\wedge w_d) = \lambda\mu\, w_1\wedge\cdots\wedge w_d$

$\Longrightarrow \varphi(\lambda l)\varphi(\mu l)+\varphi(\mu l)\varphi(\lambda l) = 2\lambda\mu = 2\omega(\lambda l, \mu l)$

② if $w \in W\oplus W'$, note that

$\varphi(w)$ changes parity of an element

$\varphi(w)\varphi(\lambda l) + \varphi(\lambda l)\varphi(w) = 0 = 2\omega(w, \lambda l)$

$\Longrightarrow$ get homomorphism $\varphi: C(V)\to \mathrm{End}(\Lambda W)$.

Also need $\varphi': C(V)\to \mathrm{End}(\Lambda W')$, defined similarly: reverse roles of W, W', $\varphi(l)$ modified by $(-1)^n$ (omit check)

$\Longrightarrow$ homomorphism $\Phi: C(V)\to \mathrm{End}(\Lambda W)\times \mathrm{End}(\Lambda W')$.

Need to check Φ injective. Use idea similar to even case. (omit details)

Now consider action of $C^+(V)$ on ΛW via φ.

Need to show $\varphi: C^+(V) \to \text{End}(\Lambda W)$ is injective

Argument very similar to even case. (omit details).

<u>Lemma</u>. ① If $x \in C^+(V)$ and $xv = vx$ for all $v \in V$, then $x \in C^0(V)$ is a scalar.

② If $x \in C^-(V)$, and $xv = -vx$ for all $v \in V$, then $x = 0$.