Math 251C, Lecture 3

Note Title 4/3/2020

Question from last time: Does choice of Borel subgroup determine maximal torus?

No: consider $GL_2\mathbb{C}$

Basis e_1, e_2 $\quad B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $T = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$

Basis e_1, e_1+e_2 $\quad B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. $T = \begin{pmatrix} x & y-x \\ 0 & y \end{pmatrix}$

If $t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ preserves $\langle e_1 \rangle$, then $c=0$

If preserves $\langle e_1+e_2 \rangle$, then $t \cdot (e_1+e_2) = ae_1 + be_1 + de_2$

$\Rightarrow a+b=d$

<u>Thm</u> ① Every f.dim representation of $GL(V)$ contains a highest weight vector.

② If representation is irreducible, any two hw vectors agree up to scalar multiple.

③ The weight of a hw vector satisfies

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \quad (n = \dim V)$$

④ For every $\mu_1 \geq \cdots \geq \mu_n$ integers, $\exists$ an irreducible representation whose hw vector has weight μ.
Furthermore, any 2 irreducible representations w/ same highest weight are isomorphic.
⑤ Representation is polynomial $\Leftrightarrow$ $\mu_n \geq 0$

Rmks $\mu_1 \geq \cdots \geq \mu_n$ are dominant weights.
This Thm is special case of a general thm usually phrased for semisimple Lie groups/algebras
$GL(V)$ not semisimple since it has a 1-dim center (scalar matrices) but
$SL(V) = \{g \in GL(V) \mid \det(g) = 1\}$ is semisimple
Dominant weights for $SL(V)$ are elements of $\mathbb{Z}^n / \mathbb{Z} \cdot (1,1,\ldots,1)$ satisfying $\mu_1 \geq \cdots \geq \mu_n$.
Note: $(1,1,\ldots,1) \longleftrightarrow$ det rep which is trivial for $SL(V)$.

Note: tensoring a representation of $GL(V)$ by $\det^d$ changes all weights of rep by adding $(d, \dots, d)$ $\Longrightarrow$

Prop If W is rational representation, then $\exists\, d$ s.t $W \otimes \det^d$ is polynomial.

Pf (not using previous thm)

Pick a basis for W, V. Then entries of $\rho: GL_n\mathbb{C} \to GL(W)$ are rational functions in variables x_{ij} (entries of $GL_n\mathbb{C}$) of the form $\frac{a(x)}{b(x)}$ where $b(g) \neq 0$ $\forall$ invertible matrices g.

Enough to show that $b(x) = \alpha \det(x)^d$ for some scalar α

Then we can clear denominators by tensoring w/ $\det^d$. We will prove this by induction on $\deg b$.

Base case $\deg b = 0$. $\Rightarrow$ b constant, done.

Induction step: $\mathbb{C}$ is algebraically closed, so $b(x)$ has a solution on space of all matrices. By assumption, $b(g) \neq 0$ for invertible g, so all solutions of $b(x) = 0$ must be non-invertible.

So, If $b(x) = 0$ Then $\det(x) = 0$. By next lemma, $\det$ is an irreducible polynomial.

$\Rightarrow$ $\det$ must divide b.

$\Rightarrow$ $b = b' \cdot \det$ where $\deg b' = \deg b - n$

If g invertible, $b'(g) = \dfrac{b(g)}{\det(g)} \neq 0$

By induction, $b' =$ Scalar $\cdot$ (power of det). $\square$

<u>Lemma</u> As a polynomial in variables X_{ij}, $\det$ is irreducible.

<u>Pf</u>. Suppose we can factor $\det = \alpha\beta$ for polynomials α, β.

$\det$ is a degree 1 polynomial in each variable separately.

For every x_{ij}, either α or β is degree 1 in x_{ij} (but not both)

Now consider x_{ij} & $x_{k,j}$ at same time.

No terms of det have both variables,

$\Rightarrow$ If α is degree 1 in x_{ij}, then it is also degree 1 in x_{kj} (and vice versa)

Same thing for x_{ij} & x_{il}.

$\Rightarrow$ If α is degree 1 in any variable, then β must be constant. $\square$

Other remarks:

Representations of $GL_n\mathbb{C} \times GL_m\mathbb{C}$ behave similarly

Borel subgroup $\rightsquigarrow$ $B \times B'$, B = upper-triangular in $GL_n\mathbb{C}$

B' = upper-triangular in $GL_m\mathbb{C}$

maximal torus $\rightsquigarrow$ $T \times T'$ T = diagonal in $GL_n\mathbb{C}$

T' = diagonal in $GL_m\mathbb{C}$

weights are pairs of sequences $(\mu_1, \ldots \mu_n), (\mu_1', \ldots, \mu_m')$

Dominant means $\mu_1 \geq \dots \geq \mu_n$

$\mu_1' \geq \dots \geq \mu_m'$

Irreducible reps $\longleftrightarrow$ dominant weights

<u>Notation</u>: $S_\lambda(\mathbb{C}^n)$ = irreducible rep of $GL_n\mathbb{C}$ of highest weight $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$

$S_\lambda\mathbb{C}^n \otimes S_\mu\mathbb{C}^m$ = irreducible rep of $GL_n\mathbb{C} \times GL_m\mathbb{C}$ of highest weight (λ, μ).

<u>Partitions</u> A partition of a non-negative integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ s.t. $\lambda_1 + \dots + \lambda_k = n$

Partitions of length k $\longleftrightarrow$ irreducible polynomial reps of $GL_k(\mathbb{C})$.

<u>Convention</u>: 0 entries of partitions can be ignored or added. e.g. $(2,2) \longleftrightarrow (2,2,0,0,0,0)$ (for reps of $GL_6\mathbb{C}$)

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$\mu_1' \geq \dots \geq \mu_n'$

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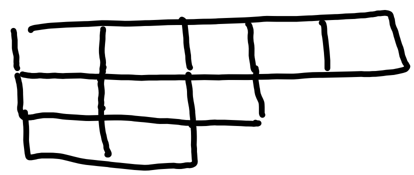
$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ (size)

$\ell(\lambda) = \#$ of λ_i which are nonzero (length)

exponential notation: $(a^b) = (\underbrace{a, a, \ldots, a}_{b \text{ times}})$

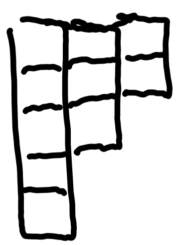
Young diagrams: represent partitions by a collection of boxes.

Eg. $\lambda = (5,3,2)$  $= Y(\lambda)$.

Given λ, have transpose partition λ^T given by

$\lambda^T_i = \#\{ j \mid \lambda_j \geq i \}$

$Y(\lambda^T) = Y(\lambda)$ flipped across diagonal

Eg. $Y((5,3,2)^T) =$  $= Y(3,3,2,1,1)$

$\nu \subseteq \lambda$ means $\nu_i \leq \lambda_i$ for all i.

$\iff Y(\nu) \subseteq Y(\lambda)$

λ/ν will mean the collection of boxes

$Y(\lambda) \setminus Y(\nu)$ (skew shape)