Math 251C, Lecture 5

Note Title 4/6/2020

<u>Thm</u> Let X be an affine variety in a rep of $GL(V)$. Let $B \subset GL(V)$ Borel subgroup.

Assume: $\exists$ B-orbit $Y \subset X$ which is dense. <u>Then</u>

(a) $\mathbb{C}[X]$ is multiplicity-free rep.

(b) Let λ be a h.w. of h.w. vector in $\mathbb{C}[X]$.

Pick $u \in Y$ and let
$$H = \mathrm{stab}(u) = \{g \in GL(V) \mid h \cdot u = u\}$$

Then, $\lambda(h) = 1 \quad \forall\, h \in H \cap B$.

<u>Pf of</u>. (a) Pick nonzero h.w. vectors $f, g \in \mathbb{C}[X]$ weight λ. Pick $u \in Y$, and $b \in B$.

$$f(b \cdot u) = (b^{-1} f)(u) = \lambda(b^{-1}) f(u)$$

So, if $f(u) = 0$, then $f(u') = 0 \quad \forall u' \in Y$

$$\Rightarrow \quad Z(f) \supset Y \underset{Y \text{ dense}}{\Rightarrow} Z(f) = X \Rightarrow f = 0$$

We conclude $f(u) \neq 0$ if $f \neq 0$

Similarly, $g(u) \neq 0$.

So $\exists \alpha \in \mathbb{C}$ so that $g(u) = \alpha f(u)$

Note: $g - \alpha f$ is a B-eigenvector of weight λ

but, $(g-\alpha f)(u) = 0. \Rightarrow g - \alpha f \equiv 0$

$\Rightarrow$ h.w. vectors in $\mathbb{C}[X]$ are unique up to scalar

(b) Pick $h \in H \cap B$. Then

$$f(u) = f(h \cdot u) = \lambda(h^{-1}) f(u)$$

Since $f(u) \neq 0, \Rightarrow \lambda(h^{-1}) = 1$

$H \cap B$ is a group, h arbitrary $\Rightarrow \lambda(h) = 1 \ \forall h \in H \cap B$. $\square$

<u>Prop</u>. X = irreducible affine variety in rep of $GL(V)$.
Assume λ, μ are weights of highest weight vectors in $\mathbb{C}[X]$. Then so is $\lambda + \mu$.

<u>Pf</u>. Pick $f, g \in \mathbb{C}[X]$ nonzero h.w. vectors of weights λ, μ respectively. <u>Claim</u>: $fg \neq 0$.

If not, $X = Z(f) \cup Z(g) \xRightarrow{X \text{ irred}} X = Z(f)$ or $X = Z(g)$

$\Rightarrow f = 0$ or $g = 0 \rightarrow\!\leftarrow$

For $b \in B$, then $b \cdot (fg) = (b \cdot f)(b \cdot g) = \lambda(b) \mu(b) fg$

λ, μ are exponents $= (\lambda + \mu)(b) fg$ $\square$

Example 1. Generic matrices.

Pick $m \geq n$ integers, consider action of $GL_n\mathbb{C} \times GL_m\mathbb{C}$ on $U = (\mathbb{C}^n \otimes \mathbb{C}^m)^*$ (n x m matrices) via

$$\underset{GL_n \times GL_m}{\underset{\cap}{(g, h)}} \cdot \underset{U}{\underset{\cap}{u}} := (g^{-1})^T u h^{-1}$$

$B \subset GL_n\mathbb{C}$, $B' \subset GL_m\mathbb{C}$ upper-triangular matrices

We have coordinates x_{ij} $i=1,\dots,n$, $j=1,\dots,m$ for U

$$U \ni u = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$$

Let A_i = upper left $i \times i$ matrix of u, $f_i = \det A_i \in \mathbb{C}[U]$

Prop. f_i is a hw vector of weight $(\underbrace{1,\dots,1}_{i}, 0,\dots,0), (\underbrace{1,\dots,1}_{i}, 0,\dots,0)$

Pf. Pick upper-triangular matrices $g \in B$, $h \in B'$.

Write $g = \begin{pmatrix} X_1 & Y_1 \\ 0 & Z_1 \end{pmatrix}$, $h = \begin{pmatrix} X_2 & Y_2 \\ 0 & Z_2 \end{pmatrix}$

where X_1, X_2 are size $i \times i$

$$((g,h)\cdot f_i)(u) = f_i((g^{-1},h^{-1})\cdot u)$$
$$= f_i(g^T u h) = \det(x_1^T A_i x_2)$$
$$= \det(x_1)\det(x_2)\, f_i(u)$$

$\Rightarrow$ f_i is h.w. vector

To compute weight, take g, h to be diagonal.

$\det(x_1)$ = product of first i entries of g

$\det(x_2)$ = ——————————— h

$\Rightarrow$ weight is $(1,\dots,1,0,\dots,0),(1,\dots,1,0\dots0)$ $\square$

Let $J = \begin{pmatrix} \underbrace{\begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{matrix}}_{n} & \underbrace{\begin{matrix} \\ 0 \\ \\ \end{matrix}}_{m-n} \end{pmatrix} \in U$

Prop. The $B \times B'$ orbit of J is open and dense.

Pf Let Y = orbit of J.

Claim $Y = \{u \mid f_i(u) \neq 0 \text{ for } i=1,\dots,n\}$

First consider $n=m$. Then J = id matrix

$Y = \{LU \mid L$ is lower triangular, U is upper triangular, both invertible$\}$

We do induction on n. (to prove $\supseteq$ of claim)

$n=1$ obvious

Assume $n>1$, true for $n-1$.

Let A be a matrix s.t. $f_i(A)\neq 0$ for $i=1..n$.

Write $A = \begin{pmatrix} A' & b \\ c & d \end{pmatrix}$ where $A' = (n-1)\times(n-1)$.

Note: $f_i(A) = f_i(A')$ $\forall i = 1, \dots, n-1$

$\Longrightarrow$ $A' = L'U'$ where L' lower-tri, invertible, U' upper-tri

$$\begin{pmatrix} A' & b \\ c & d \end{pmatrix} = \overset{L}{\overset{\shortparallel}{\begin{pmatrix} L' & 0 \\ x & 1 \end{pmatrix}}} \overset{U}{\overset{\shortparallel}{\begin{pmatrix} U' & y \\ 0 & z \end{pmatrix}}}$$

where $x = c\underbrace{(U')^{-1}}_{\text{invertible}}$, $y = (L')^{-1}b$, $z = d - xy$.

$$\begin{pmatrix} U' & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} L' & 0 \\ x & 1 \end{pmatrix}^{-1} \underbrace{\begin{pmatrix} A' & b \\ c & d \end{pmatrix}}_{\text{invertible b/c } f_n(A)\neq 0} \Longrightarrow \begin{pmatrix} U' & y \\ 0 & z \end{pmatrix} \text{ invertible}$$

If $m>n$, write $A = (A' \ \ A'')$, A' $n\times n$

By previous case $A' = LU$

$$A = LJ\begin{pmatrix} u & L^{-1}\text{“}A\text{''} \\ 0 & id_{m-n}\end{pmatrix}$$ Prove claim

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$$Y = (U \setminus Z(f_1)) \cap (U \setminus Z(f_2)) \cap \dots \cap (U \setminus Z(f_n))$$

open . . .

finite intersection of open is open $\Rightarrow$ Y is open.

Since U is irreducible, open & nonempty $\Rightarrow$ dense $\square$

<u>Cor</u>. $\mathbb{C}[U]$ is multiplicity free.

<u>Lemma</u>. If (λ, λ') is weight of h.w. vector in $\mathbb{C}[U]$, then $\lambda_i = \lambda_i'$ for $i=1,\dots,n$ and $\lambda_j' = 0$ for $j>n$. Also, $\lambda_n \geq 0$.

<u>Pf</u>. $\operatorname{stab}(J) \ni \left(\begin{pmatrix} x_1^{-1} & & 0 \\ & \ddots & \\ 0 & & x_n^{-1}\end{pmatrix}, \begin{pmatrix} x_1 & & & & \\ & \ddots & & & \\ & & x_n & & \\ & & & \ddots & \\ & & & & x_m\end{pmatrix}\right) = (g,h)$

where $x_1 \dots x_m \neq 0$.

$$(\lambda,\lambda')(g,h) = \lambda(g)\lambda'(h) = x_1^{-\lambda_1} \cdots x_n^{-\lambda_n} \cdot x_1^{\lambda_1'} \cdots x_m^{\lambda_m'}$$

$$= x_1^{\lambda_1' - \lambda_1} \cdots x_n^{\lambda_n' - \lambda_n} x_{n+1}^{\lambda_{n+1}'} \cdots x_m^{\lambda_m'}$$

By Thm, this $=1$ if (λ, λ') is weight of a h.w. vector $\Longrightarrow$ all exponents must be 0

$$\Longrightarrow \lambda_1 = \lambda_1', \ldots, \lambda_n = \lambda_n'$$
$$\lambda_{n+1}' = \cdots = \lambda_m' = 0$$

To see $\lambda_n \geq 0$, note that all weights of U^* are non-negative (hence true for all powers of U^*) $\square$