Math 251C, Lecture 8

Note Title 4/15/2020

Last time: $S_\lambda V$ has basis of weight vectors indexed by semistandard Young tableaux (SSYT)

Formula for #SSYT

① $$\dim S_\lambda \mathbb{C}^n = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

② (Hook-content formula)

$$\dim S_\lambda \mathbb{C}^n = \prod_{u \in Y(\lambda)} \frac{n + c(u)}{h(u)}$$

→ $\dim S_\lambda \mathbb{C}^n$ is polynomial in n of degree $|\lambda|$.

boxes in Young diagram

$c(u) = j - i$ where j = column index of u
i = row index of u

<u>content</u>

$h(u)$ = # boxes in same row of u appearing to the right of u
+ # boxes in same column of u appearing below u
+1

<u>hook length</u>

Ex. $\lambda = (6,3,1)$

content

$$\begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 4 & 5 \\ \hline -1 & 0 & 1 \\ \cline{1-3} -2 \\ \cline{1-1} \end{array}$$

hook length

$$\begin{array}{|c|c|c|c|c|c|} \hline 8 & 6 & 5 & 3 & 2 & 1 \\ \hline 4 & 2 & 1 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array}$$

Ex $\lambda = (3,2)$. $\dim S_{32}\mathbb{C}^n$:

① $i=1$: $$\frac{3-2+1}{1}\cdot\frac{3+2}{2}\cdot\frac{3+3}{3}\cdots\frac{3+(n-1)}{n-1}$$

$$= 2\,\frac{(n+2)!}{4!}\cdot\frac{1}{(n-1)!}$$

$i=2$: $$\frac{2+1}{1}\cdot\frac{2+2}{2}\cdots\frac{2+(n-2)}{n-2} = \frac{n!}{2\,(n-2)!}$$

$$\dim S_{32}\mathbb{C}^n = \cancel{2}\,\frac{(n+2)!}{4!\,(n-1)!}\cdot\frac{n!}{\cancel{2}(n-2)!}$$

$$= \frac{n(n+2)(n+1)(n)(n-1)}{4!}$$

← polynomial in n of degree 5

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② c $\quad$ h

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline -1 & 0 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|} \hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2} \end{array}$$

$$\dim S_{32}\mathbb{C}^n = \frac{n(n+1)(n+2)(n-1)\,n}{4\cdot 3\cdot 1\cdot 2\cdot 1}$$

$$= \frac{n^2(n+1)(n+2)(n-1)}{4!}$$

Symmetric polynomials/functions

Lemma. Let ρ be rep of $GL_n\mathbb{C}$. Then $(\operatorname{char}\rho)(x_1, \dots x_n)$ is symmetric, i.e., $\forall \sigma \in \mathfrak{S}_n$

$$(\operatorname{char}\rho)(x_1 \dots x_n) = (\operatorname{char}\rho)(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Pf. $\sigma \rightsquigarrow$ permutation matrix $M(\sigma)$ corresponding to linear map $e_i \longrightarrow e_{\sigma(i)}$

$$M(\sigma)^{-1}\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} M(\sigma) = \begin{pmatrix} x_{\sigma(1)} & & \\ & \ddots & \\ & & x_{\sigma(n)} \end{pmatrix}$$

$$(\operatorname{char}\rho)(x_1, \dots, x_n) = \operatorname{trace}\rho\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$$

& trace is invariant under conjugation. $\square$

Recall: for every rep ρ, $\exists\, d$ s.t. $\rho \otimes \det^d$ is polynomial. Character of polynomial rep is a polynomial in $x_1, \dots x_n$.

$$\Longrightarrow \operatorname{char}\rho(x_1 \dots x_n) = \frac{f(x_1 \dots x_n)}{(x_1 \cdots x_n)^d}$$

f is a polynomial

Let $\Lambda(n)$ = set of symmetric polynomials in $x_1, \ldots, x_n$ w/ $\mathbb{Z}$-coefficients

$\Lambda(n)$ is a ring under usual addition/multiplication which contains $\mathrm{char}\,\rho$ for any polynomial rep ρ

For polynomial reps of $GL_n\mathbb{C} \times GL_m\mathbb{C}$, let

$\Lambda(n,m)$ = polynomials w/ $\mathbb{Z}$-coefficients in $x_1, \ldots, x_n, y_1, \ldots y_m$ which are symmetric in x's & y's

$\Lambda(n,m)$ contains characters of polynomial reps of $GL_n\mathbb{C} \times GL_m\mathbb{C}$.

<u>Def</u>. The <u>Schur polynomial</u> $s_\lambda(x_1, \ldots x_n)$ is character of $S_\lambda \mathbb{C}^n$.

$$s_\lambda(x_1, \ldots x_n) = \sum_{\substack{\text{SSYT } T \\ \text{of shape } \lambda}} x^{\mu(T)}$$

<u>General facts</u>

① $\{s_\lambda(x) \mid \ell(\lambda) \leq n\}$ is a basis for $\Lambda(n)$

② Weyl character formula

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, define

$$a_\alpha = \det\left(x_i^{\alpha_j}\right)_{i,j=1,\dots,n} = \det \begin{pmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots & x_1^{\alpha_n} \\ & \vdots & & \\ x_n^{\alpha_1} & \cdots & & x_n^{\alpha_n} \end{pmatrix}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, x^{\sigma(\alpha)}$$

Define $\rho = (n-1, n-2, \dots, 1, 0) \in \mathbb{Z}_{\geq 0}^n$

<u>Thm.</u> (WCF)

$$s_\lambda(x_1 \dots x_n) = \frac{a_{\lambda+\rho}}{a_\rho} = \frac{\det\left(x_i^{\lambda_j + n - j}\right)_{i,j=1\dots n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}$$

$$= \frac{\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, x^{\sigma(\lambda+\rho)}}{\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, x^{\sigma(\rho)}}.$$

Plethysm

Given a rep W of $GL_n\mathbb{C}$, we can construct $S_\lambda(W)$ ← think of as a rep of $GL_n\mathbb{C}$

Let $w_1, \dots w_N$ be weights of W (repeating if appear w/ multiplicity)

$$\Longrightarrow (\text{char } S_\lambda W)(x_1,\dots,x_n) = s_\lambda(x^{w_1}, x^{w_2}, \dots, x^{w_N})$$

$$= s_\lambda \circ \text{char} W$$

↑ plethysm

can also define $f \circ \text{char} W$ for any sym. polynomial f in N vars:

$$f = \sum_\lambda c_\lambda s_\lambda \ ; \quad f \circ \text{char} W = \sum_\lambda c_\lambda (s_\lambda \circ \text{char} W)$$

Question: given λ, μ we can define $S_\lambda(S_\mu(\mathbb{C}^n))$. How does it decompose into irreducible reps? $\cong \bigoplus_\nu (S_\nu \mathbb{C}^n)^{\oplus p^\nu_{\lambda\mu}}$

↖ How to determine?

Very hard outside special cases.

Introduce new variable t, we will work w/ power series in t w/ coefficients in $\Lambda(n)$.

$h_d(x) = \mathrm{char}\, \mathrm{Sym}^d \mathbb{C}^n = s_d(x)$

<u>Lemma</u>

$\Lambda(n)[\![t]\!] \ni$ $$\sum_{d \geq 0} h_d(x_1 \ldots x_n)\, t^d = \frac{1}{\prod_{i=1}^{n}(1 - x_i t)}$$

<u>Pf</u>. LHS is sum of $x_1^{d_1} \cdots x_n^{d_n} t^{d_1 + \cdots + d_n}$ varying over all choices of $d_i \geq 0$

RHS, using geometric series is $\prod_{i=1}^{n}\left(\sum_{d_i \geq 0} x_i^{d_i} t^{d_i}\right)$

which is the same $\square$

<u>Thm</u> (Cauchy identity) in $\Lambda(n,m)[\![t]\!]$:

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(1 - x_i y_j t)^{-1} = \sum_{\lambda} s_\lambda(x_1 \ldots x_n)\, s_\lambda(y_1, \ldots, y_m)\, t^{|\lambda|}$$

sum over all $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0)$

$r = \min(n, m)$