Math 202B, Winter 2020
Homework 1
Due: January 27 in class

Please do not look up solutions directly online. You are free to work with other students, but solutions must be written in your own words. Please cite any sources that you use or any people you collaborated with.

(1) Let G be a finite group and let V, W be finite-dimensional G-representations. Define $\Phi\colon V^* \otimes W \to \mathrm{Hom}(V, W)$ by $\Phi(\sum_i f_i \otimes w_i) = F$ where $F(v) = \sum_i f_i(v) w_i$. Show that Φ is well-defined and is a G-equivariant isomorphism.

(2) Let G be a finite abelian group and let V be an irreducible representation over an algebraically closed field (of arbitrary characteristic). Use Schur's lemma to prove that $\dim V = 1$.

(3) Let G be a group. Define $[G, G]$ to be the subgroup of G generated by elements of the form $xyx^{-1}y^{-1}$ where $x, y \in G$.
   (a) Show that $[G, G]$ is a normal subgroup and that $G/[G, G]$ is abelian.
   (b) Show that $[G, G]$ is in the kernel of any representation $\rho\colon G \to \mathbf{GL}(V)$ where $\dim(V) = 1$ and deduce that there is a bijection between the 1-dimensional representations of G and of $G/[G, G]$.

(4) Let X be a set with G-action and let $V = \mathbf{C}[X]$ be the permutation representation. Let χ_1 be the character of the trivial representation.
   (a) Show that (χ_V, χ_1) is the number of orbits of G acting on X.
   (b) For the rest of the problem, assume that X has size at least 2 and that G has 1 orbit on X.
   The line spanned by $\sum_{x \in X} e_x$ is a subrepresentation, let U be a subrepresentation of $\mathbf{C}[X]$ which is a complement of it. Show that $(\chi_U, \chi_1) = 0$.
   (c) Define an action of G on $X \times X$ by $g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2)$. Show that $\chi_{\mathbf{C}[X \times X]} = \chi_V^2$.
   (d) Show that U is irreducible if and only if G has exactly 2 orbits on $X \times X$.

(5) Let $\mathbf{F}$ be a field, let $G = \mathbf{GL}_2(\mathbf{F})$ be the group of invertible 2×2 matrices with entries in $\mathbf{F}$, and let X be the set of lines, i.e., 1-dimensional subspaces in $\mathbf{F}^2$ which has a natural action of G. Show that $X \times X$ has exactly 2 orbits. When $\mathbf{F}$ is finite, the representation U from above is called the **Steinberg representation** of G.

(6) Let $n > 1$ and let $\mathbf{k}$ be a field. Prove that $\{(x_1, \dots, x_n) \in \mathbf{k}^n \mid x_1 + \cdots + x_n = 0\}$ is an irreducible representation of the symmetric group $\mathfrak{S}_n$ when $\mathbf{k}$ has characteristic 0. Show that this remains true if $\mathbf{k}$ has characteristic $p > 0$ and p does not divide n. What happens when p divides n?