Math 202B, Winter 2020
Homework 2
Due: February 5 in class

Please do not look up solutions directly online. You are free to work with other students, but solutions must be written in your own words. Please cite any sources that you use or any people you collaborated with.

For all problems below, except #6, assume that all representations are over the complex numbers.

(1) Let G_1, G_2 be finite groups and consider representations over $\mathbf{C}$ (the results below extend to other situations, but this is simplest using what we've learned).
  (a) If V, W are irreducible representations of G_1, G_2, respectively, then $V \boxtimes W$ is an irreducible representation of $G_1 \times G_2$.
  (b) Let $V_1, \dots, V_n$ and $W_1, \dots, W_m$ be complete lists of irreducible representations (up to isomorphism) of G_1 and G_2, respectively. Then $\{V_i \boxtimes W_j\}$ is a complete list of irreducible representations (up to isomorphism) of $G_1 \times G_2$.

(2) Compute the character table for the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. The product is determined by $i^2 = j^2 = k^2 = ijk = -1$ and requiring -1 to commute with everything.

(3) Let p be a prime and let G be the group (under multiplication) of upper-triangular 3×3 matrices whose entries are in $\mathbf{Z}/p$ and whose diagonal entries are all 1:

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| a, b, c \in \mathbf{Z}/p \right\}.$$

  (a) Determine the size of $G/[G,G]$.
  (b) Let V be the set of functions $f \colon \mathbf{Z}/p \to \mathbf{C}$. This is a complex vector space under addition of functions and scalar multiplication by complex numbers. Pick $\omega \in \mathbf{C}$ such that $\omega^p = 1$. Show that there is a unique homomorphism $\rho_\omega \colon G \to \mathbf{GL}(V)$ that satisfies

$$(\rho_\omega \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} f)(x) = f(x-1), \qquad (\rho_\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} f)(x) = \omega^x f(x).$$

  (c) Show that ρ_ω is irreducible if $\omega \neq 1$ and that ρ_ω is not isomorphic to $\rho_{\omega'}$ if $\omega \neq \omega'$.
  (d) Classify all of the irreducible representations of G.

(4) Let G be a finite group and let $H \subseteq G$ be a subgroup of index 2.
  (a) Show that H is a normal subgroup and hence is a union of conjugacy classes of G.
  (b) Let γ be a conjugacy class of G which is contained in H. Show that either γ is still a single conjugacy class of H, or that it is a union of two conjugacy classes of H, which both have the same size.
  (Recall that the size of the conjugacy class of $x \in G$ is $|G/C_G(x)|$ where $C_G(x) = \{h \in G \mid hx = xh\}$ and consider $C_G(x) \cap H$.)

(c) Let V be an irreducible representation of G. Show that $(\chi_V, \chi_V)_H \leq 2$. Deduce that the restriction of G to H is either irreducible, or a direct sum of two non-isomorphic irreducible representations.

(d) In the second case of (c), suppose that $V|_H \cong V_1 \oplus V_2$ for irreducible representations V_1, V_2 of H. Pick $g \in G \setminus H$. Show that $g \cdot V_1 := \{g \cdot v \mid v \in V_1\}$ is an H-subrepresentation of V and that in fact $g \cdot V_1 = V_2$. Conclude that $\dim V_1 = \dim V_2$.

(5) Compute the character table of the alternating group $\mathfrak{A}_4 \subset \mathfrak{S}_4$.

(6) Let G be a finite group and $H \subseteq G$ be a subgroup. Let $\mathbf{k}$ be a field. Let $\rho\colon H \to \mathbf{GL}(W)$ be a representation. Define

$$F_H^G(W) = \{f\colon G \to W \mid f(xh) = \rho(h^{-1})(f(x)) \text{ for all } x \in G,\ h \in H\}.$$

(a) Show that $F_H^G(W)$ is a G-representation with the action $(g \cdot f)(x) = f(g^{-1}x)$.

(b) Pick coset representatives $g_1, \ldots, g_r$ for G/H. Define

$$\Phi\colon F_H^G(W) \to \mathrm{Ind}_H^G(W) \qquad \Phi(f) = \sum_{i=1}^{r} e_{g_i} \otimes f(g_i).$$

Show that Φ is a G-equivariant isomorphism.