Math 202B, Winter 2020
Homework 3
Due: February 19 in class

Please do not look up solutions directly online. You are free to work with other students, but solutions must be written in your own words. Please cite any sources that you use or any people you collaborated with.

#7 is meant to show you the basic properties and operations of formal power series. It is strictly speaking not necessary to know how these are proven, but we will use them in lecture. So don't turn in solutions for #7.

(1) Given any finite-dimensional vector space V with a basis $v_1, \dots, v_m$ and a symmetric bilinear form $\langle , \rangle$, prove that the dimension of $V/V^\perp$ is the rank of the Gram matrix $(\langle v_i, v_j \rangle)_{i,j=1,\dots,m}$.

(2) Find a formula (and prove it) for the number of standard Young tableaux for the following families of partitions:
  (a) $(n, 1^k)$ for $n \geq 1$
  (b) $(n, 2)$
  (c) (challenge) (n, n)

(3) Show that $\mathbf{M}^{(n-2,2)} \cong \mathbf{S}^{(n-2,2)} \oplus \mathbf{S}^{(n-1,1)} \oplus \mathbf{S}^{(n)}$ over a field of characteristic 0.

(4) Write the polytabloid e_t for $t = \begin{matrix} 4 & 1 \\ 3 & 2 \end{matrix}$ as a linear combination of standard polytabloids.

(5) Let p be a prime dividing n and let $\mathbf{k}$ be a field of characteristic p. Let $U = \{(x_1, \dots, x_n) \in \mathbf{k}^n \mid x_1 + \cdots + x_n = 0\}$ and let L be the line spanned by $(1, 1, \dots, 1)$. Show that U/L is an irreducible $\mathfrak{S}_n$-representation.

(6) For this exercise, our field is of characteristic 0.
  (a) Show that the sign representation of $\mathfrak{S}_n$ is isomorphic to the Specht module $\mathbf{S}^{(1^n)}$.
  (b) Prove that for any finite group G, given an irreducible representation V and a 1-dimensional representation W, the tensor product $V \otimes W$ is also irreducible.
  (c) Combining the above, we see that $\mathbf{S}^\lambda \otimes \mathbf{S}^{(1^n)}$ is irreducible, so must be isomorphic to a Specht module. We will now describe which one.
  Let $\mu = \lambda^\dagger$ be the transpose partition. Pick a λ-tableau t and let $t^\dagger$ be the $\lambda^\dagger$-tableau obtained by flipping the values across the diagonal. Let $R_{t^\dagger}$ denote the row-stabilizer of $t^\dagger$, i.e., the set of $\sigma \in \mathfrak{S}_n$ such that $\{\sigma t^\dagger\} = \{t^\dagger\}$ and let $\rho_{t^\dagger} = \sum_{\sigma \in R_{t^\dagger}} \sigma$.
  Let $u \in \mathbf{S}^{(1^n)}$ be a nonzero element and recall that every $\lambda^\dagger$-tabloid is of the form $\{\sigma t^\dagger\}$ for some $\sigma \in \mathfrak{S}_n$ (though the choice is not unique). Show that there is a well-defined $\mathfrak{S}_n$-equivariant linear map
  $$\varphi \colon \mathbf{M}^{\lambda^\dagger} \to \mathbf{S}^\lambda \otimes \mathbf{S}^{(1^n)}, \qquad \varphi(\{\sigma t^\dagger\}) = \sigma \rho_{t^\dagger}(\{t\} \otimes u).$$
  (d) Show that $\varphi(e_{t^\dagger}) \neq 0$. Conclude that $\mathbf{S}^{\lambda^\dagger} \cong \mathbf{S}^\lambda \otimes \mathbf{S}^{(1^n)}$.
  (Hint: First show that $\varphi(e_{t^\dagger}) = (\rho_t \kappa_t \{t\}) \otimes u$. Then show that $\langle \rho_t \kappa_t \{t\}, \{t\} \rangle \neq 0$ where $\langle , \rangle$ is the pairing on $\mathbf{M}^\lambda$.)

(7) (don't turn in) This exercise is to familiarize you with formal power series rings. Let R be a commutative ring with multiplicative identity 1. Let $R[\![t]\!]$ denote the set of formal linear combinations $\sum_{n\ge 0} r_n t^n = r_0 + r_1 t + \cdots$. These are called formal power series. Two are equal if and only if the coefficients are all pairwise equal. We define $[t^n]\sum_{n\ge 0} r_n t^n = r_n$.

(a) Show that $R[\![t]\!]$ is a commutative ring with the operations

$$\sum_{n\ge 0} r_n t^n + \sum_{n\ge 0} s_n t^n = \sum_{n\ge 0}(r_n + s_n)t^n$$

$$(\sum_{n\ge 0} r_n t^n)(\sum_{n\ge 0} s_n t^n) = \sum_{n\ge 0}(\sum_{i=0}^{n} r_i s_{n-i})t^n.$$

(b) Show that $\sum_{n\ge 0} r_n t^n$ has a multiplicative inverse if and only if r_0 has a multiplicative inverse in R. (If r_0 is invertible, the identity $(\sum_{n\ge 0} r_n t^n)(\sum_{n\ge 0} s_n t^n) = 1$ gives a system of equations for the s_n which can be solved by induction.)

(c) If $F_1(t), F_2(t), \dots$ are formal power series, we say that the sequence converges to $F(t)$ if for each n, we have $[t^n]F_i(t) = [t^n]F(t)$ for all but finitely many i. In that case we write $\lim_{i\to\infty} F_i(t) = F(t)$.

This allows us to define infinite sums as limits of partial sums, i.e., $\sum_{i=1}^{\infty} F_i(t) = \lim_{j\to\infty}\sum_{i=1}^{j} F_i(t)$ when it exists and also infinite products $\prod_{i=1}^{\infty} F_i(t) = \lim_{j\to\infty}\prod_{i=1}^{j} F_i(t)$. Show that an infinite sum, if it exists, can be rearranged arbitrarily and give the same value. Show the same for infinite products.

If R also has a notion of convergence, then we can take that into account and define the limit to be $F(t) = \sum_{n\ge 0}(\lim_{i\to\infty}[t^n]F_i(t))\, t^n$. This is relevant when R is the ring of bounded degree power series where convergence is defined by asking that the coefficient of each term is constant for $i \gg 0$. Show that the rearrangement properties continue to hold for this example.

(d) Let $F(t) = \sum_{n\ge 0} f_n t^n$ and $G(t)$ be formal power series such that $G(t)$ has no constant term. Define the composition to be $F(G(t)) = \sum_{n\ge 0} f_n G(t)^n$. Explain why this is well-defined. We define $\exp(G(t)) = F(G(t))$ where $F(t) = \sum_{n\ge 0} t^n/n!$. Show that $\exp(\sum_{i\ge 0} F_i(t)) = \prod_{i\ge 0}\exp(F_i(t))$ assuming that $\sum_{i\ge 0} F_i(t)$ is well-defined.

(e) We define derivatives of formal power series in the obvious way: $F'(t) = \sum_{n\ge 1} n f_n t^{n-1}$. All of the familiar rules hold (chain rule, product rule, etc.) Many of the familiar Taylor series from calculus have analogues in the world of formal power series, and for the most part they behave exactly like they do in calculus (e.g., log is the compositional inverse of exp).