Math 202B, Winter 2020
Homework 4
Due: March 2 in class

Please do not look up solutions directly online. You are free to work with other students, but solutions must be written in your own words. Please cite any sources that you use or any people you collaborated with.

(1) (a) Consider the permutation $\sigma = 45723618 \in \mathfrak{S}_8$, i.e., $\sigma(1) = 4$, $\sigma(2) = 5$, etc. Interpret it as a matrix by putting a 1 in position $(i, \sigma(i))$ for $i = 1, \dots, 8$ and 0 elsewhere. Apply RSK to this matrix.
    (b) Find a permutation such that RSK produces SYT of shape $(4, 2, 1, 1)$. In general, find a permutation such that RSK produces SYT of shape λ for any partition λ.

(2) Show that

$$s_\lambda(x_1, y_1, x_2, y_2, \dots) = \sum_{\mu \subseteq \lambda} s_\mu(x_1, x_2, \dots) s_{\lambda/\mu}(y_1, y_2, \dots).$$

(3) Let $\lambda = (2n-2, 2n-4, \dots, 4, 2, 0)$. Find a simple factorization for $s_\lambda(x_1, \dots, x_n)$.

(4) Simplify the product

$$\left(\sum_{n \ge 0} e_n t^n\right) \left(\sum_{\lambda} s_{2\lambda} t^{2|\lambda|}\right)$$

where $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$ and the second sum is over all partitions.

(5) Explain how to deduce the identity

$$s_\nu h_\mu = \sum_\lambda K_{\lambda/\nu,\mu} s_\lambda$$

using just Pieri's rule (the special case where $\ell(\mu) = 1$).

(6) Let λ, μ, ν be partitions of n. Let χ^λ be the character of the Specht module $\mathbf{S}^\lambda$. The product $\chi^\lambda \chi^\mu$ (naive product, not induction product!) is the character of $\mathbf{S}^\lambda \otimes \mathbf{S}^\mu$, and we can decompose it:

$$\chi^\lambda \chi^\mu = \sum_\nu g^\nu_{\lambda,\mu} \chi^\nu$$

for some non-negative integers g.
    (a) Show that g is invariant under permutations of ν, λ, μ, i.e.,

$$g^\nu_{\lambda,\mu} = g^\nu_{\mu,\lambda} = g^\lambda_{\nu,\mu} = g^\lambda_{\mu,\nu} = g^\mu_{\lambda,\nu} = g^\mu_{\nu,\lambda}.$$

    For that reason, we write them more symmetrically as $g_{\lambda,\mu,\nu}$.
    (b) Using the Frobenius characteristic ch, we can transfer the product of characters to symmetric functions: given $f, g \in \Lambda_{\mathbf{Q}}$, define

$$f * g = \mathrm{ch}(\mathrm{ch}^{-1}(f)\mathrm{ch}^{-1}(g)).$$

    For any class function χ, show that $\mathrm{ch}(\chi) * p_\lambda = \chi(\lambda) p_\lambda$.
    (c) We showed that $p_1, p_2, \dots$ are algebraically independent and hence every $f \in \Lambda_{\mathbf{Q}}$ can be written as a polynomial in the p_n with $\mathbf{Q}$-coefficients. Define $\frac{\partial}{\partial p_n} f$ to be partial derivative of this polynomial where we are treating the p_n as variables,

so for example, $\frac{\partial}{\partial p_2}(p_2^2 p_5 + p_1 + p_1 p_2^3) = 2p_2 p_5 + 3p_1 p_2^2$. Show that for all $f, g \in \Lambda$, we have

$$\left\langle n \frac{\partial}{\partial p_n} f, g \right\rangle = \langle f, p_n g \rangle.$$

(d) Let Y^1 be the character of the permutation representation of $\mathfrak{S}_n$ on $\mathbf{C}^n$. Show that $\mathrm{ch}(Y^1) * s_\lambda = s_1 s_{\lambda/1}$.
You may use that $\mathrm{ch}(\chi^\lambda) = s_\lambda$ (we will prove this soon in class).
Hint: first show that $\mathrm{ch}(Y^1) * p_\lambda = m_1(\lambda) p_\lambda = p_1 \frac{\partial}{\partial p_1} p_\lambda$ and show that $\frac{\partial}{\partial p_1} s_\lambda = s_{\lambda/1}$ by showing they pair the same way against all symmetric functions.

(e) Use $Y^1 = \chi^{(n-1,1)} + \chi^n$ to deduce a formula for $g_{(n-1,1),\lambda,\mu}$.