NOTES FOR MATH 202B

STEVEN V SAM

The goal of this course is to discuss some generalities on complex linear representation theory of finite groups, the theory of symmetric functions, and how this connects with the symmetric groups. At the end we'll discuss a bit about explicit constructions of irreducible representations of symmetric groups and polynomial functors. Some references are [Se] for representation theory, [J] and [FH] for symmetric groups in particular, [Sta] and [Mac] for symmetric functions.

CONTENTS

1. Linear representations of finite groups

1.1. **Definitions.** Let G be a finite group. The identity element will be called 1_G. A **(linear) representation** of G over a field $\mathbf{k}$ is a homomorphism

$$\rho_V \colon G \to \mathbf{GL}(V)$$

for some $\mathbf{k}$-vector space V, where $\mathbf{GL}(V)$ is the group of invertible linear operators on V. Equivalently, giving a representation is the same as giving a linear action of G on V, i.e., a function $G \times V \to V$ which we think of as a multiplication $g \cdot v$ for $g \in G$ and $v \in V$ such that:

- $g \cdot (v + v') = g \cdot v + g \cdot v'$,
- $(gg') \cdot v = g \cdot (g' \cdot v)$,
- $1_G \cdot v = v$, and
- $g \cdot (\lambda v) = \lambda(g \cdot v)$ for any $\lambda \in \mathbf{k}$.

The multiplication is obtained by setting $g \cdot v = \rho_V(g)(v)$. We will always assume that V is finite-dimensional.

We will generally take the perspective that V "is" the representation, and the information ρ_V is implicit but not always mentioned. So properties of a representation such as dimension, being nonzero, etc. come from the vector space V.

Let V and V' be two representations of G. A linear map $f \colon V \to V'$ is G-**equivariant** if for all $g \in G$, we have

$$f \circ \rho_V(g) = \rho_{V'}(g) \circ f,$$

or more compactly: $f(g \cdot v) = g \cdot f(v)$ for all $v \in V$. An **isomorphism** is a G-equivariant map which is invertible; if an isomorphism exists we write $V \cong V'$.

Example 1.1.1.   (1) For any vector space V we can define $\rho_V(g)$ to be the identity on V. This is clearly a representation. When $\dim V = 1$, this is called the **trivial representation**.

    (2) Let X be a finite set with a G-action. Recall this means that we have a function $G \times X \to X$ denoted $(g, x) \mapsto g \cdot x$ such that $1_G \cdot x = x$ for all $x \in X$, and $g \cdot (g' \cdot x) = (gg') \cdot x$ for all $g, g' \in G$ and $x \in X$. Let $V = \mathbf{k}[X]$ be the $\mathbf{k}$-vector space

with basis $\{e_x \mid x \in X\}$ and define ρ_V by $\rho_V(g)e_x = e_{g \cdot x}$. This is the **permutation representation** of X.

A special case of a group action is when $X = G$ and $g \cdot x = gx$ is given by the group operation. In that case, $\mathbf{k}[G]$ is called the **regular representation**. $\square$

Lemma 1.1.2. *All of the eigenvalues of $\rho(g)$ are roots of unity.*

If $\mathbf{k}$ is algebraically closed and has characteristic 0, then $\rho(g)$ is diagonalizable for all $g \in G$.

Proof. Let λ be an eigenvalue of $\rho(g)$ with eigenvector v. Then $\rho(g)^{|G|} = 1$ but also $\rho(g)^{|G|}v = \lambda^{|G|}v$, so $\lambda^{|G|} = 1$.

Consider the Jordan normal form of $\rho(g)$, which recall is an upper-triangular matrix whose diagonal entries are the eigenvalues of $\rho(g)$ and whose superdiagonal (the entries in positions $(i, i+1)$) are either 0 or 1. Then $\rho(g)$ is diagonalizable if and only if the superdiagonal is 0. Furthermore, if any of those entries are 1, then no positive power of $\rho(g)$ is equal to the identity, but we know that $\rho(g)^{|G|} = 1$. $\square$

Remark 1.1.3. The second part can fail in positive characteristic. For instance, let $G = \mathbf{Z}/2 = \{1, z\}$ and $\mathbf{k}$ be a field of characteristic 2. Then $\rho(z) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ defines a representation (since $\rho(z)^2$ is the identity) but $\rho(z)$ is not diagonalizable. $\square$

1.2. **Basic operations.** Let V and W be representations of G. There are a few basic operations we will make use of:

- (Direct sum) The direct sum $V \oplus W$ is a representation with multiplication given by $g \cdot (v, w) = (g \cdot v, g \cdot w)$.
- (Dual) Recall that the dual space V^* is the vector space of linear functionals $V \to \mathbf{k}$. It is a representation with multiplication given as follows: if $f \in V^*$, then $g \cdot f$ is the functional defined by $(g \cdot f)(v) = f(g^{-1} \cdot v)$.
- (Tensor product) Recall that the tensor product $V \otimes W$ is a vector space which is spanned by symbols of the form $v \otimes w$ with $v \in V$ and $w \in W$ subject to the relations
  - $(v + v') \otimes w = v \otimes w + v' \otimes w$,
  - $v \otimes (w + w') = v \otimes w + v \otimes w'$,
  - $\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$ for any $\lambda \in \mathbf{k}$.
  Then $V \otimes W$ is a representation of G via $g \cdot \sum_i (v_i \otimes w_i) = \sum_i (g \cdot v_i) \otimes (g \cdot w_i)$.
- (Hom spaces) We let $\mathrm{Hom}(V, W)$ denote the vector space of linear maps $V \to W$. This is a representation of G via $(g \cdot f)(v) = g \cdot f(g^{-1} \cdot v)$. Here we are using both actions: $g^{-1} \cdot v$ is the multiplication for V and the other $\cdot$ is the multiplication for W.
- (Invariants) $V^G = \{v \in V \mid g \cdot v = v \text{ for all } g \in G\}$ is the space of G-invariants and is clearly a subrepresentation of V.

When V and W are finite-dimensional, we have a natural isomorphism $V^* \otimes W \to \mathrm{Hom}(V, W)$ given by $\sum_i f_i \otimes w_i \mapsto F$ where $F(v) = \sum_i f_i(v)w_i$. Furthermore, this is a G-equivariant isomorphism (exercise).

It follows from the definitions that $\mathrm{Hom}(V, W)^G$ is the space of G-equivariant maps $V \to W$. We will usually denote this by $\mathrm{Hom}_G(V, W)$.

1.3. **Irreducible representations.** Recall the following from linear algebra: if V is a vector space with a subspace W, then a complement of W is another subspace W' of V such that

$W \cap W' = 0$ and $W + W' = V$. In that case, we write $V = W \oplus W'$. Complements always exist but are not unique if $W \neq V$ and $W \neq 0$. The data of a complement is equivalent to a projection $\pi \colon V \to V$ (i.e., a linear map satisfying $\pi^2 = \pi$) whose image is W: given a projection, we define $W' = \ker \pi$; on the other hand, given W', every vector v can be written uniquely as $v_1 + v_2$ where $v_1 \in W$ and $v_2 \in W'$, and we define $\pi(v) = v_1$.

A **subrepresentation** of a representation V is a subspace $W \subseteq V$ such that $\rho_V(g)w \in W$ whenever $g \in G$ and $w \in W$. We can then define a homomorphism $\rho_W \colon G \to \mathbf{GL}(W)$ by $\rho_W(g) = \rho_V(g)|_W$.

If $f \colon V \to V'$ is G-equivariant, then $\ker f$ is a subrepresentation of V and image f is a subrepresentation of V'.

Lemma 1.3.1. *Suppose that the characteristic of* $\mathbf{k}$ *is either* 0 *or is* $p > 0$ *and that* p *does not divide* $|G|$. *Then given a subrepresentation* $W \subseteq V$, *there exists a subrepresentation* $U \subseteq V$ *which is a complement of* W, *i.e.,* $V = W \oplus U$.

Proof. First pick an arbitrary complement W' of W. This gives a projection $\pi \colon V \to V$. Define a new linear map $\psi \colon V \to V$ by

$$\psi = \frac{1}{|G|} \sum_{g \in G} \rho_V(g) \circ \pi \circ \rho_V(g)^{-1}.$$

Note that $\rho_V(h) \circ \psi \circ \rho_V(h)^{-1} = \psi$ for any $h \in G$: by the above formula we have

$$\rho_V(h) \circ \psi \circ \rho_V(h)^{-1} = \frac{1}{|G|} \sum_{g \in G} \rho_V(h) \circ \rho_V(g) \circ \pi \circ \rho_V(g)^{-1} \rho_V(h)^{-1}$$

$$= \frac{1}{|G|} \sum_{g \in G} \rho_V(hg) \circ \pi \circ \rho_V(hg)^{-1}$$

The last sum is just the sum for ψ except indexed differently: do the change of variables $g \mapsto h^{-1}g$. In particular, this means that ψ is G-equivariant.

Next, we claim that $\psi^2 = \psi$ and has image W. Pick $v \in V$. Then $\pi(\rho_V(g)^{-1}v) \in W$ since π has image W. Since W is a subrepresentation, $\rho_V(g)(\pi(\rho_V(g)^{-1}v)) \in W$ as well, so $\psi(v)$ is a sum of elements in W, and hence belongs to W. If $v \in W$, then $\rho_V(g)^{-1}v \in W$, and so $\pi(\rho_V(g)^{-1}v) = \rho_V(g)^{-1}v$ since π^2 is the identity on W. This gives

$$\psi(v) = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(\rho_V(g)^{-1}(v)) = \frac{1}{|G|}|G|v = v.$$

So we take $U = \ker \psi$ and conclude that U is another complement W. Since ψ is G-equivariant, U is a subrepresentation of V. $\qquad\square$

A nonzero representation V is **irreducible** if it has no nonzero subrepresentations other than itself. A representation which is not irreducible is called **reducible**.

Example 1.3.2. Any 1-dimensional representation is automatically irreducible. An example of a reducible representation would be to take $V = \mathbf{k}^2$ and have all g act by the identity. In that case, any 1-dimensional subspace of V is a nonzero subrepresentation. This example also shows that the complement found above is not unique: for any 1-dimensional subrepresentation of V, any other 1-dimensional subspace gives a complement which is also a subrepresentation. $\qquad\square$

Theorem 1.3.3 (Schur's lemma)**.** *Let V and W be nonzero irreducible representations of G and let $\varphi\colon V \to W$ be G-equivariant.*

(1) *Either φ is an isomorphism, or $\varphi = 0$.*
(2) *Suppose $\mathbf{k}$ is algebraically closed. If $V = W$, then φ is a scalar multiple of the identity.*

Proof. (1) $\ker \varphi$ is a subrepresentation of V. Since V is irreducible, this means that $\ker \varphi = 0$ or $\ker \varphi = V$. In the second case, $\varphi = 0$. In the first case, φ is injective and so the image of φ is nonzero. Since W is irreducible and image φ is a subrepresentation, this means that image $\varphi = W$. So φ is also surjective, so we conclude that φ is an isomorphism.

(2) Let λ be an eigenvalue of φ. Then $\varphi - \lambda \cdot \mathrm{id}_V$ is G-equivariant, and has nonzero kernel. In particular, it must be the 0 map by (a). $\square$

Example 1.3.4. Take $\mathbf{k} = \mathbf{R}$ the field of real numbers and $G = \mathbf{Z}/4 = \{1, z, z^2, z^3\}$ and $V = \mathbf{R}^2$. Then $\rho(z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ uniquely determines a representation (since $\rho(z)^4$ is the identity). Furthermore, this is an irreducible representation (any 1-dimensional subrepresentation must be an eigenspace for $\rho(z)$ but they are not realizable over $\mathbf{R}$). Since G is abelian, $\rho(z)$ commutes with all $\rho(g)$ and so in particular, $\rho(z)\colon V \to V$ is a G-equivariant map which is not a scalar multiple of the identity. The problem is that if we extend our scalars to $\mathbf{C}$, then V is no longer irreducible. $\square$

The previous example prompts a definition: we say that V is **absolutely irreducible** if it remains irreducible upon enlarging the field of coefficients. Actually it suffices to know that it remains irreducible upon enlarging to the algebraic closure $\overline{\mathbf{k}}$ of $\mathbf{k}$. We won't go much into the details of subtleties of non-algebraically closed fields, though.

Theorem 1.3.5 (Maschke)**.** *Suppose that the characteristic of $\mathbf{k}$ is either 0 or is $p > 0$ and that p does not divide $|G|$. Every finite-dimensional representation of G is a direct sum of irreducible subrepresentations.*

Furthermore, this decomposition is unique in the following sense: if $V = V_1 \oplus \cdots \oplus V_r$ and $V = V_1' \oplus \cdots \oplus V_s'$ are two decompositions of V into irreducible subrepresentations, then $r = s$ and there exists a permutation σ of $1, \ldots, r$ so that $V_i \cong V_{\sigma(i)}'$ for all i.

The empty direct sum is the 0 vector space, so in the proof, we assume that $\dim V > 0$.

Proof. Induction on dimension of V. If $\dim V = 1$, then V must be irreducible. Otherwise, if $\dim V > 1$ and V is not irreducible, pick a nonzero subrepresentation $W \subset V$ with $W \neq V$. By the previous result, we can find a subrepresentation $U \subset V$ so that $V = W \oplus U$. But $\dim W < \dim V$ and $\dim U < \dim V$, so by induction, both W, U are direct sums of irreducible subrepresentations.

For the second statement, consider the identity map on V. By Schur's lemma, the component maps $V_i \to V_j'$ of the identity are either 0 or isomorphisms. Let $W_1, \ldots, W_a$ be all of the V_i which are isomorphic to V_1 and let $W_1', \ldots, W_b'$ be all of the V_j' which are isomorphic to V_1. We conclude that the restriction of the identity to $W_1 \oplus \cdots \oplus W_a \to W_1' \oplus \cdots \oplus W_b'$ must be an isomorphism since the W_i have 0 maps to all other V_j'. This means that $a = b$; remove these summands from V and repeat to get the desired conclusion. $\square$

1.4. **Characters.** Now we assume that $\mathbf{k} = \mathbf{C}$.

The **character** of ρ is the function $\chi_\rho \colon G \to \mathbf{C}$ defined by $\chi_\rho(g) = \text{Tr}(\rho(g))$ where Tr denotes the trace of a linear operator. This is constant on conjugacy classes of G:

$$\chi_\rho(hgh^{-1}) = \text{Tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \text{Tr}(\rho(g)) = \chi_\rho(g).$$

We'll also write χ_V if V is the vector space of the representation. Since $\rho(1_G)$ is the identity, we have $\chi_\rho(1_G) = \dim V$.

We'll use the following fact many times: the trace of a linear operator is the sum of its eigenvalues.

Recall that an **algebraic integer** is a complex number α which is the solution to a monic polynomial with integer coefficients, i.e., there exist integers $c_0, \ldots, c_{n-1}$ so that $\alpha^n + \sum_{i=0}^{n-1} c_i \alpha^i = 0$. The following results are standard results in algebra, and we will not repeat their proofs:

Proposition 1.4.1. (1) *The set of algebraic integers is a subring of the complex numbers: they are closed under addition, subtraction, and multiplication.*
(2) *Every rational number which is an algebraic integer is in fact an integer.*

An important example of algebraic integers are roots of unity: these are the solutions to the equations $z^n - 1 = 0$ for some n.

Note that algebraic integers could be imaginary, for example, $\sqrt{-1}$ is an algebraic integer.

Proposition 1.4.2. *For any $g \in G$ and representation V, $\chi_V(g)$ is an algebraic integer.*

Proof. All of the eigenvalues of $\rho(g)$ are roots of unity, which are algebraic integers. Since $\chi_V(g)$ is the sum of these eigenvalues, it is also an algebraic integer. $\square$

Lemma 1.4.3. *Let V be a representation with character χ_V. Then $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ and the character of its dual is given by $\chi_{V^*}(g) = \overline{\chi_V(g)}$ where the bar denotes complex conjugation.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\rho_V(g)$ with eigenvectors $v_1, \ldots, v_n$. Let $f_1, \ldots, f_n$ be the dual basis for V^*. Then f_i is an eigenvector with eigenvalue λ_i^{-1} for $\rho_{V^*}(g)$ (recall the action is through g^{-1}). Since the λ_i are roots of unity, we have $\lambda_i^{-1} = \overline{\lambda_i}$ (recall that $\lambda\overline{\lambda} = |\lambda|$ for any complex number and roots of unity have absolute value 1). In particular, $\chi_{V^*}(g) = \chi_V(g^{-1}) = \lambda_1^{-1} + \cdots + \lambda_n^{-1} = \overline{\lambda_1 + \cdots + \lambda_n} = \overline{\rho_V(g)}$. $\square$

Lemma 1.4.4. *Let V, W be representations. The character of $V \oplus W$ is given by $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$.*

Proof. Let $v_1, \ldots, v_n$ be an eigenbasis for $\rho_V(g)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and let $w_1, \ldots, w_m$ be an eigenbasis for $\rho_W(g)$ with eigenvalues $\mu_1, \ldots, \mu_m$. Then $\{(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)\}$ is an eigenbasis for $V \oplus W$ with eigenvalues $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m$. $\square$

Lemma 1.4.5. *Let V, W be representations. The character of $V \otimes W$ is given by $\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g)$.*

Proof. Let $v_1, \ldots, v_n$ be an eigenbasis for $\rho_V(g)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and let $w_1, \ldots, w_m$ be an eigenbasis for $\rho_W(g)$ with eigenvalues $\mu_1, \ldots, \mu_m$. Then $\{v_i \otimes w_j\}_{i,j}$ is an eigenbasis for $\rho_{V \otimes W}(g)$ with eigenvalues $\{\lambda_i\mu_j\}$. So $\chi_{V \otimes W}(g) = \sum_{i,j} \lambda_i\mu_j = (\sum_i \lambda_i)(\sum_j \mu_j) = \chi_V(g)\chi_W(g)$. $\square$

Lemma 1.4.6. *Let $V = \mathbf{C}[X]$ be a permutation representation on the set X. Then $\chi_V(g) = |\{x \in X \mid g \cdot x = x\}|$.*

Proof. The elements of X give a basis for $\mathbf{C}[X]$. In matrix form $\rho(g)$ becomes a permutation matrix and the number of 1's on the diagonal is just the number of elements fixed by g. $\square$

Lemma 1.4.7 (Projection formula)**.** *Define* $\varphi \colon V \to V$ *by*

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot v.$$

φ *is a projection and its image is* V^G. *In particular,* $\dim V^G = \mathrm{Tr}(\varphi)$.

Proof. First we prove the image is contained in V^G: for any $h \in G$, we have

$$h \cdot \varphi(v) = \frac{1}{|G|} \sum_{g \in G} hg \cdot v = \frac{1}{|G|} \sum_{g \in G} g \cdot v = \varphi(v).$$

where in the second equality, we reindexed the sum since $\{hg \mid g \in G\} = \{g \mid g \in G\}$.

On the other hand, given $w \in V^G$, we have $\varphi(w) = w$, and so the image is all of V^G.

These two facts imply that φ is a projection: $\varphi^2(v) = \varphi(\varphi(v))$ and $\varphi(v) \in V^G$ which implies that $\varphi^2(v) = \varphi(v)$.

For the last statement, note that the eigenvalues of a projection are either 0 or 1 (since it's a root of the polynomial $t^2 - t$) and the multiplicity of 1 is its rank. $\square$

A function $G \to \mathbf{C}$ which is constant on conjugacy classes is called a **class function** and $\mathrm{CF}(G)$ is the set of class functions $G \to \mathbf{C}$. This is a $\mathbf{C}$-vector space. Define a pairing on $\mathrm{CF}(G)$ by:

$$(\varphi, \psi)_G = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\psi(g)}$$

where the overline means complex conjugation. If we don't need to specify G, we'll just write $(,)$.

Lemma 1.4.8. $(,)$ *is an inner product on* $\mathrm{CF}(G)$.

Proof. Let c be the number of conjugacy classes of G and order the conjugacy classes as $\gamma_1, \ldots, \gamma_c$. Then $\mathrm{CF}(G) \cong \mathbf{C}^c$ by sending a class function f to $(\sqrt{|\gamma_1|}f(\gamma_1), \ldots, \sqrt{|\gamma_c|}f(\gamma_c))$. Under this isomorphism, $(,)$ is $1/|G|$ times the standard inner product on $\mathbf{C}^c$, and hence is itself an inner product. $\square$

Lemma 1.4.9. *Given two representations* V, W, *we have* $\dim \mathrm{Hom}_G(V, W) = (\chi_V, \chi_W)$.

Proof. We have $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G \cong (V^* \otimes W)^G$. The character of $V^* \otimes W$ is given by $\chi_{V^* \otimes W}(g) = \overline{\chi_V(g)}\chi_W(g)$. From the projection formula, for any representation U, we have $\dim U^G = \frac{1}{|G|} \sum_{g \in G} \chi_U(g)$. Applying this to $U = V^* \otimes W$, we get

$$\dim(V^* \otimes W)^G = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\chi_W(g) = (\chi_W, \chi_V) = \overline{(\chi_V, \chi_W)}.$$

But note that $\dim(V^* \otimes W)^G$ is an integer, so in particular, $\overline{(\chi_V, \chi_W)} = (\chi_V, \chi_W)$. $\square$

A priori, characters are complex-valued. For some special groups, the values never take imaginary values.

Proposition 1.4.10. *Suppose that* g *is conjugate to* g^{-1} *for all* $g \in G$. *Then* $\chi_V(g)$ *is a real number for all* $g \in G$ *and all representations* V.

Proof. $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, but if they are conjugate, we also have $\chi_V(g) = \chi_V(g^{-1})$, so $\chi_V(g) \in \mathbf{R}$. $\qquad\square$

Proposition 1.4.11. *Suppose that for each integer m that is relatively prime to $|G|$, and for all $g \in G$, we have that g is conjugate to g^m. Then $\chi_V(g)$ is an integer for all $g \in G$ and all representations V.*

Proof. Since $\chi_V(g)$ is an algebraic integer, it will suffice to show that it is a rational number by Proposition 1.4.1. The proof is similar to the previous result, but uses a little bit of Galois theory: let L be the field generated by $\mathbf{Q}$ and a primitive $|G|$th root of unity ω. For every m that is relatively prime to $|G|$, there is an automorphism σ_m of L determined by replacing ω by ω^m, and an element of L is in $\mathbf{Q}$ if and only if it is fixed by all of these automorphisms.

From the previous results, we know that $\chi_V(g) \in L$ for all $g \in G$ and all representations V. Furthermore, $\sigma_m(\chi_V(g)) = \chi_V(g^m)$ since the trace of an mth power of a linear operator is the sum of the mth powers of its eigenvalues, so our assumption together with the Galois theory above implies that $\chi_V(g) \in \mathbf{Q}$ for all $g \in G$ and representations V. $\qquad\square$

1.5. **Classification of representations.**

Lemma 1.5.1. *Let $\rho\colon G \to \mathbf{GL}(V)$ be a representation and $f \in \mathrm{CF}(G)$ a class function. Define*

$$\rho_f = \sum_{g \in G} f(g)\rho(g).$$

If V is irreducible, then $\rho_f = \lambda \cdot \mathrm{id}_V$ where

$$\lambda = \frac{|G|}{\dim V}(f, \overline{\chi_V})_G.$$

Proof. Pick $h \in G$. Then

$$\rho(h)\rho_f\rho(h)^{-1} = \sum_{g \in G} f(g)\rho(hgh^{-1}) = \sum_{g \in G} f(hgh^{-1})\rho(hgh^{-1}) = \rho_f$$

where the second equality follows from $f \in \mathrm{CF}(G)$ and the third equality follows since conjugation of h is a permutation of the elements of G. This implies that ρ_f commutes with all $h \in G$ and hence is a scalar by Schur's lemma.

To determine λ, we have

$$\dim(V)\lambda = \mathrm{Tr}(\rho_f) = \sum_{g \in G} f(g)\,\mathrm{Tr}(\rho(g)) = \sum_{g \in G} f(g)\chi_V(g) = |G|(f, \overline{\chi_V})_G. \qquad\square$$

Theorem 1.5.2.   • *The characters of the irreducible representations form an orthonormal basis for $\mathrm{CF}(G)$. In particular, the number of isomorphism classes of irreducible representations of G is equal to the number of conjugacy classes of G.*
   • *If two representations have the same character, then they are isomorphic.*

Proof. Let V, W be irreducible. By Schur's lemma, $\dim \mathrm{Hom}_G(V, W)$ is 0 if $V \not\cong W$ and is 1 if $V \cong W$. In particular, $(\chi_V, \chi_W) = 0$ if $V \not\cong W$ and is 1 if $V \cong W$. So if $V_1, V_2, \ldots$ are pairwise non-isomorphic irreducible representations, then $\chi_{V_1}, \chi_{V_2} \ldots$ are orthonormal. In particular, they are linearly independent. Next, we need to show that they span $\mathrm{CF}(G)$. Let f be a function in the orthogonal complement of the span of the χ_{V_i}. By Lemma 1.5.1, ρ_f (notation used there) is 0 for all irreducible ρ, and hence for all representations ρ since

Maschke's theorem implies every representation is a direct sum of irreducibles. Now consider the regular representation $\mathbf{C}[G]$ of G. In that case we have

$$0 = \rho_f(e_{1_G}) = \sum_{g \in G} f(g) e_g.$$

Since the e_g are a basis, this implies that $f(g) = 0$ for all $g \in G$, i.e., that $f = 0$. So $\mathrm{CF}(G)$ is spanned by the χ_{V_i}.

Since the dimension of the space of class functions is the number of conjugacy classes of G, we see that this is also the number of irreducible representations of G. Let $V_1, \ldots, V_c$ be a complete list of irreducible representations of G up to isomorphism.

Next, given a representation V, we have $V \cong V_1^{\oplus m_1} \oplus \cdots \oplus V_c^{\oplus m_c}$ for some non-negative integers $m_1, \ldots, m_c$ (the multiplicities). Again, by Schur's lemma and the previous result, $m_i = (\chi_{V_i}, \chi_V)$. If W is another representation, it is isomorphic to V if and only if the corresponding multiplicities agree with $m_1, \ldots, m_c$. Hence if $\chi_V = \chi_W$, then $V \cong W$. $\square$

Corollary 1.5.3. *The multiplicity of an irreducible representation V in the regular representation $\mathbf{C}[G]$ is $\dim V$.*

Proof. The multiplicity is given by $(\chi_V, \chi_{\mathbf{C}[G]})$. Note that $\chi_{\mathbf{C}[G]}(1_G) = |G|$ and $\chi_{\mathbf{C}[G]}(g) = 0$ for $g \neq 1_G$. In particular,

$$(\chi_V, \chi_{\mathbf{C}[G]}) = \frac{1}{|G|} \chi_V(1_G)|G| = \chi_V(1_G) = \dim V. \qquad \square$$

Corollary 1.5.4. *Let $V_1, \ldots, V_c$ be the irreducible representations of G with dimensions $d_1, \ldots, d_c$. Then $d_1^2 + \cdots + d_c^2 = |G|$.*

Proof. The dimension of $\mathbf{C}[G]$ is $|G|$ and it contains V_i with multiplicity d_i, so $|G| = \sum_{i=1}^c d_i^2$. $\square$

Corollary 1.5.5. *If G is abelian, Then all irreducible representations are 1-dimensional.*

Proof. The conjugacy classes of abelian groups are singletons, so there are $c = |G|$ many conjugacy classes. The only solution to $c = d_1^2 + \cdots + d_c^2$ where the d_i are positive integers are $d_1 = \cdots = d_c = 1$. $\square$

1.6. **Examples.**

1.6.1. *Direct products.* Suppose we are given two groups G_1, G_2 and representations ρ_1, ρ_2 on vector spaces V_1, V_2. Then $G_1 \times G_2$ has a linear action on $V_1 \otimes V_2$ by

$$(g_1, g_2) \cdot \sum_i v(1)_i \otimes v(2)_i = \sum_i g_1 \cdot v(1)_i \otimes g_2 \cdot v(2)_i.$$

Hence we get a representation $\rho_1 \otimes \rho_2$ of $G_1 \times G_2$. Its character is given by

$$\chi_{\rho_1 \otimes \rho_2}(g_1, g_2) = \chi_{\rho_1}(g_1) \chi_{\rho_2}(g_2).$$

This is the **external tensor product** of representations. To emphasize that it is a representation of the direct product, we will write $V_1 \boxtimes V_2$. Here are some facts which are left as exercises:

(1) If V, W are irreducible, then so is $V \boxtimes W$.
(2) Let $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$ be complete lists of irreducible representations (up to isomorphism) of G_1 and G_2, respectively. Then $\{V_i \boxtimes W_j\}$ is a complete list of irreducible representations (up to isomorphism) of $G_1 \times G_2$.

1.6.2. *Abelian groups.* We saw in Corollary 1.5.5 that all irreducible representations of abelian groups are 1-dimensional. We will completely describe them starting with cyclic groups $G = \mathbf{Z}/m$. Let ω be a primitive mth root of unity, i.e., $\omega^m = 1$ but $\omega^n \neq 1$ for all $n < m$. For a concrete example, we can take $\exp(2\pi i/m)$. For $0 \leq i \leq m-1$, define a homomorphism $\rho_i \colon \mathbf{Z}/m \to \mathbf{GL}_1(\mathbf{C})$ by $\rho_i(g) = \omega^{ig}$. These are all irreducible representations (since they are 1-dimensional), they are not isomorphic to each other (their characters are all different), and there are m of them, which is the number of conjugacy classes of $\mathbf{Z}/m$, so we have described all of them.

A general finite abelian group is isomorphic to a direct product of $\mathbf{Z}/m$ for various m, so we can construct all of the irreducible representations using the previous example.

1.6.3. *Dihedral groups.* For $n \geq 3$, let D_n be the symmetries of a regular n-gon. Then $|D_n| = 2n$ and we will use without proof that when n is odd, there are $(n+3)/2$ conjugacy classes, and when n is even, there are $(n+6)/2$ conjugacy classes.

Actually, if we center the regular n-gon at the origin in the plane, then each element of D_n (rotation or reflection) is a linear operator, so we get a representation $\rho \colon D_n \to \mathbf{GL}_2(\mathbf{R})$, which is called the reflection representation. We can then extend the coefficients to $\mathbf{C}$; I'll leave it as an exercise to show that the result is an irreducible representation. There is also a 1-dimensional representation which sends g to the determinant of $\rho(g)$ under the reflection representation. We call this the sign representation.

For concreteness, consider D_5, which has 4 conjugacy classes and size 10. So it has 4 irreducible representations, let $d_1, \ldots, d_4$ be their dimensions. We know that $d_1^2 + d_2^2 + d_3^2 + d_4^2 = 10$, and so we must have $d_i \leq 2$ for all i. The only solution is $\{1, 1, 2, 2\}$. We know three examples: the trivial representation, the sign representation, and the reflection representation. Since the reflection representation is real-valued, it is isomorphic to its dual. So that doesn't create a new representation. Furthermore, we can create a 2-dimensional representation by tensoring the reflection representation with the sign representation, is it new? Turns out it is not – but we will stop with any further calculations.

1.6.4. *Symmetric groups.* A large portion of this course will be devoted to working out the characters of the symmetric groups $\mathfrak{S}_n$. We will content ourselves now with some basic facts and small examples. First, $|\mathfrak{S}_n| = n!$. The conjugacy classes are easy to describe:

Lemma 1.6.1. *Two permutations are conjugate if and only if they have the same cycle type decomposition. Hence, the conjugacy classes of $\mathfrak{S}_n$ are naturally indexed by partitions of n, and the number of irreducible representations of $\mathfrak{S}_n$ is the partition number $p(n)$*

Proof. if $(i_1, i_2, \ldots, i_k)$ denotes the cycle $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_k \mapsto i_1$, then we can use the identity

$$\tau(i_1, i_2, \ldots, i_k)\tau^{-1} = (\tau(i_1), \tau(i_2), \ldots, \tau(i_k)). \qquad \square$$

Every character is integer-valued: this amounts to showing that for every permutation σ, and every m coprime to $n!$, then σ^m and σ have the same cycle type. Note that m being coprime to $n!$ means it is coprime to every $i = 1, \ldots, n$. If $c_1 \cdots c_r = \sigma$ is the decomposition into disjoint cycles, then $\sigma^m = c_1^m \cdots c_r^m$. Furthermore, if c_i has length i and m is coprime to i, then c_i^m is again a cycle of length i. So σ and σ^m have the same cycle type and are hence conjugate to one another.

Every $\mathfrak{S}_n$ has a representation on $\mathbf{C}^n$ by sending each σ to the corresponding permutation operator: $\rho(\sigma)(e_i) = e_{\sigma(i)}$. This is the **permutation representation** of $\mathfrak{S}_n$. This is not

irreducible if $n > 1$: there are two subrepresentations, one spanned by the vector $(1, 1, \ldots, 1)$ and the other given by the subspace $\{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 0\}$. The latter is called the **standard representation** of $\mathfrak{S}_n$. It is irreducible (exercise). Furthermore, there is also a 1-dimensional **sign representation** which sends σ to its sign (which is 1 if σ is even and -1 if σ is odd; recall that a permutation is even if it is a product of an even number of transpositions, and odd otherwise).

Let's consider $\mathfrak{S}_3$. It has 3 conjugacy classes and 6 elements. We know of 3 representations already: trivial, sign, and standard, so they are all of them. Let's compute the characters (the columns are the partition which counts the cycle lengths of a permutation):

	$(1,1,1)$	$(2,1)$	(3)
trivial	1	1	1
sign	1	-1	1
standard	2	0	-1

How to compute the character of the standard? We know that trivial + standard is the permutation representation and that has character $3, 1, 0$ (the number of fixed points of a permutation by Lemma 1.4.6). Note that if we tensor sign and standard, the character is the same as standard, so we conclude that sign tensored with standard is isomorphic to standard again.

Let's also do $\mathfrak{S}_4$. It has 5 conjugacy classes and 24 elements. Again, we know of 3 representations: trivial, sign, and standard, which have dimensions $1, 1, 3$.

	$(1,1,1,1)$	$(2,1,1)$	$(3,1)$	$(2,2)$	(4)
trivial	1	1	1	1	1
sign	1	-1	1	1	-1
standard	3	1	0	-1	-1

We computed the character of standard in the same way as for $\mathfrak{S}_3$. This time, sign tensored with standard gives a different character, so we get another 3-dimensional representation. Since the sum of the squares of the dimensions is 24, the last representation must be 2-dimensional. Actually, we can figure out its character from the other 4 since the sum of the irreducible characters each multiplied by their dimension is the character of the regular representation.

	$(1,1,1,1)$	$(2,1,1)$	$(3,1)$	$(2,2)$	(4)
sign $\otimes$ standard	3	-1	0	-1	1
2$-$dim	2	0	-1	2	0

Actually we can construct that last representation as follows. Let X be the set of ways to write $\{1, 2, 3, 4\}$ as a union of two 2-element subsets, so $X = \{12|34, 13|24, 14|23\}$. Then $\mathfrak{S}_4$ acts on X and $\mathbf{C}[X]$ contains a 1-dimensional representation (the sum of all of the elements). The complement has the correct character so is the representation we're looking for.

Using these ad hoc arguments, we will quickly hit a wall, so we'll need to do something else to get a general theory for $\mathfrak{S}_n$.

1.7. **The group algebra.** Representations can be recast in the language of modules over a ring as follows. Let $\mathbf{k}$ be a field and G a group. The **group algebra**, denoted $\mathbf{k}[G]$ is the vector space with basis $\{e_g \mid g \in G\}$ and multiplication $e_g e_h = e_{gh}$. This multiplication is associative and has a unit e_{1_G}.

We will now show that representations of G over the field $\mathbf{k}$ are the same information as left $\mathbf{k}[G]$-modules. First, suppose we are given a representation $\rho\colon G \to \mathbf{GL}(V)$. We define a left $\mathbf{k}[G]$-module structure on G by $(\sum_g \alpha_g e_g)(v) = \sum_g \alpha_g \rho_V(g)v$. On the other hand, if M is a left $\mathbf{k}[G]$-module, then in particular it is a vector space over $\mathbf{k}$, and $m \mapsto e_g \cdot m$ is linear operator for all g. Let $\rho_M(g)$ denote this linear operator. Then $\rho_M\colon G \to \mathbf{GL}(M)$ is a homomorphism. We omit checking all of the small details, but these two operations are inverse to each other so we have proven:

Proposition 1.7.1. *Representations of G over $\mathbf{k}$ can be identified with left $\mathbf{k}[G]$-modules.*

Furthermore, we have a built-in notion of homomorphism of left $\mathbf{k}[G]$-modules. It is routine to check that this is the same as G-equivariant maps between representations under the above correspondence.

1.8. **Restriction and induction.** Given a subgroup $H \subseteq G$, any representation ρ of G becomes a representation of H by restricting the map. This is called the **restriction** of ρ, and is denoted $\mathrm{Res}_H^G \rho$. In fact, restriction makes sense for any class functions.

On the other hand, given a representation $\rho\colon H \to \mathbf{GL}(V)$, one can define the **induced representation** $\mathrm{Ind}_H^G V$ which is a representation of G. This is conceptually clearest to define using tensor products. First, as before, V is a left $\mathbf{k}[H]$-module. Second, $\mathbf{k}[G]$ can be made into a *right* $\mathbf{k}[H]$-module via $g \cdot h = gh$ for $g \in G$ and $h \in H$. We can then define the tensor product over $\mathbf{k}[H]$: $\mathbf{k}[G] \otimes_{\mathbf{k}[H]} V$. More generally, if R is a (not necessarily commutative) ring and M is a right R-module and N is a left R-module, then $M \otimes_R N$ is the abelian group spanned by symbols $m \otimes n$ with $m \in M$ and $n \in N$ subject to the relations:

- $(m + m') \otimes n = m \otimes n + m' \otimes n$,
- $m \otimes (n + n') = m \otimes n + m \otimes n'$,
- $vr \otimes w = v \otimes rw$ for any $r \in R$.

In general, there is no further structure on R. In our case, we can make $\mathbf{k}[G] \otimes_{\mathbf{k}[H]} V$ into a left $\mathbf{k}[G]$-module by $g \cdot (\sum_i e_{g_i} \otimes v_i) = \sum_i e_{gg_i} \otimes v_i$. Note that the tensor product will be a $\mathbf{k}$-vector space. If $v_1, \ldots, v_n$ is a basis for V and $g_1, \ldots, g_r$ are representatives for the left cosets G/H, then a basis for $\mathrm{Ind}_H^G V$ is $\{e_{g_i} \otimes v_j\}$, so in particular, $\dim(\mathrm{Ind}_H^G V) = |G/H| \dim V$.

To compute $g \cdot (e_{g_i} \otimes v_j)$, first find k so that $gg_i \in g_k H$. Then $g_j^{-1} gg_i \in H$, and we have

$$g \cdot (e_{g_i} \otimes v_j) = e_{g_k} \cdot (g_k^{-1} gg_i) \otimes v_j = e_{g_k} \otimes g_k^{-1} gg_i \cdot v_j.$$

Then express $g_k^{-1} gg_i \cdot v_j$ as a linear combination of $v_1, \ldots, v_n$ using the action of H on V.

Example 1.8.1. Let X be a set with a transitive G-action, i.e., for all $x, y \in X$, there exists $g \in G$ so that $gx = y$. Pick any point $x \in X$ and let H be its stabilizer. Then the left action of G on G/H is the same as the action of G on X under the identification $gH \mapsto gx$. Hence $\mathbf{k}[G/H] \cong \mathbf{k}[X]$. Furthermore, we can identify $\mathbf{k}[G/H]$ with $\mathrm{Ind}_H^G \mathbf{k}$ where $\mathbf{k}$ is the trivial representation of H. $\square$

This construction is transitive in the following sense: if we have 3 groups $K \subseteq H \subseteq G$, and a representation V of K, then there is a natural identification

$$\mathrm{Ind}_H^G(\mathrm{Ind}_K^H V) = \mathrm{Ind}_K^G V.$$

Proposition 1.8.2. *For $g \in G$, we have*

$$\chi_{\mathrm{Ind}_H^G V}(g) = \sum_{\substack{1 \le i \le r \\ g_i^{-1} g g_i \in H}} \chi_V(g_i^{-1} g g_i).$$

Proof. We use the basis mentioned above and compute the trace of g acting on $\mathrm{Ind}_H^G V$ with respect to it. Consider the subspace $e_{g_i} \otimes V$ which is the span of $e_{g_i} \otimes v_j$ for $j = 1, \ldots, n$. We can think of the matrix for g as a block matrix corresponding to these subspaces. In order for there to be diagonal entries in this portion of the matrix, we need $g g_i \equiv g_i \pmod{H}$, i.e., $g_i^{-1} g g_i \in H$. In that case, for $v \in V$, we have

$$g \cdot e_{g_i} \otimes v = e_{g_i} \otimes (g_i^{-1} g g_i) v,$$

so that the corresponding block matrix in the diagonal is $\rho_V(g_i^{-1} g g_i)$, and its trace is $\chi_V(g_i^{-1} g g_i)$. We sum over all of these traces to get the trace of g acting on $\mathrm{Ind}_H^G V$. $\square$

The above formula can be extended to make sense for any class function, namely, we define

$$(\mathrm{Ind}_H^G \chi)(g) = \sum_{\substack{1 \le i \le r \\ g_i^{-1} g g_i \in H}} \chi(g_i^{-1} g g_i),$$

so that we get a linear function $\mathrm{Ind}_H^G \colon \mathrm{CF}(H) \to \mathrm{CF}(G)$. The important fact is:

Theorem 1.8.3 (Frobenius reciprocity). *Given $\varphi \in \mathrm{CF}(H)$ and $\psi \in \mathrm{CF}(G)$, we have*

$$(\mathrm{Ind}_H^G \varphi, \psi)_G = (\varphi, \mathrm{Res}_H^G \psi)_H.$$

Proof. Let $g_1, \ldots, g_r$ be coset representatives for G/H. Then

$$(\mathrm{Ind}_H^G \varphi, \psi)_G = \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{1 \le i \le r \\ g_i^{-1} g g_i \in H}} \varphi(g_i^{-1} g g_i) \overline{\psi(g)} = \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{1 \le i \le r \\ g_i^{-1} g g_i \in H}} \varphi(g_i^{-1} g g_i) \overline{\psi(g_i^{-1} g g_i)}$$

where the last equality follows from conjugation-invariance of ψ. Now, the inputs are actually elements of H. How many times does a particular $h \in H$ appear, i.e., how many g, g_i satisfy $g_i^{-1} g g_i = h$? In fact, for each i, there is exactly one g since we can rewrite the relation as $g = g_i h g_i^{-1}$. Hence the sum becomes

$$\frac{r}{|G|} \sum_{h \in H} \varphi(h) \overline{\psi(h)}.$$

But $r = |G/H|$, so the above simplifies to $(\varphi, \mathrm{Res}_H^G \psi)_H$. $\square$

A proof can be given using formal properties of tensor products.

2. Constructing symmetric group representations

2.1. **Partitions.** A **partition** of a nonnegative integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k \ge 0$ and $\lambda_1 + \cdots + \lambda_k = n$. We will consider two partitions the same if their nonzero entries are the same. It will also be convenient to make the convention that $\lambda_i = 0$ whenever $i > \ell(\lambda)$. And for shorthand, we may omit the commas, so the partition $(1, 1, 1, 1)$ of 4 can be written as 1111. As a further shorthand, the exponential notation is used for repetition, so for example, 1^4 is the partition $(1, 1, 1, 1)$. We let $\mathrm{Par}(n)$ be the set

of partitions of n, and denote the size by $p(n) = |\mathrm{Par}(n)|$. By convention, $\mathrm{Par}(0)$ consists of exactly one partition, the empty one.

Example 2.1.1.

$$\mathrm{Par}(1) = \{1\},$$
$$\mathrm{Par}(2) = \{2, 1^2\},$$
$$\mathrm{Par}(3) = \{3, 21, 1^3\},$$
$$\mathrm{Par}(4) = \{4, 31, 22, 21^2, 1^4\},$$
$$\mathrm{Par}(5) = \{5, 41, 32, 31^2, 2^2 1, 21^3, 1^5\}. \qquad \square$$

If λ is a partition of n, we write $|\lambda| = n$ (**size**). Also, $\ell(\lambda)$ is the number of nonzero entries of λ (**length**). For each i, $m_i(\lambda)$ is the number of entries of λ that are equal to i.

It will often be convenient to represent partitions graphically. This is done via **Young diagrams**, which is a collection of left-justified boxes with λ_i boxes in row i.[1] For example, the Young diagram
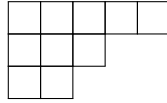
corresponds to the partition $(5, 3, 2)$. Flipping across the main diagonal gives another partition $\lambda^\dagger$, called the **transpose**. In our example, flipping gives
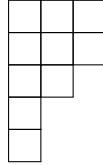
So $(5, 3, 2)^\dagger = (3, 3, 2, 1, 1)$. In other words, the role of columns and rows has been interchanged. This is an important involution of $\mathrm{Par}(n)$ which we will use later.

We will use several different partial orderings of partitions:

- $\lambda \subseteq \mu$ if $\lambda_i \le \mu_i$ for all i.
- The **dominance order**: $\lambda \le \mu$ if $\lambda_1 + \cdots + \lambda_i \le \mu_1 + \cdots + \mu_i$ for all i. Note that if $|\lambda| = |\mu|$, then $\lambda \le \mu$ if and only if $\lambda^\dagger \ge \mu^\dagger$. So transpose is an order-reversing involution on the set of partitions of a fixed size.
- The **lexicographic order**: for partitions of the same size, $\lambda \le^R \mu$ if $\lambda = \mu$, or otherwise, there exists i such that $\lambda_1 = \mu_1, \ldots, \lambda_{i-1} = \mu_{i-1}$, and $\lambda_i < \mu_i$. This is a total ordering.

The following lemma will be useful, so we isolate it here.

Lemma 2.1.2. *Let $a_{\lambda, \mu}$ be a set of integers indexed by partitions of a fixed size n. Assume that $a_{\lambda, \lambda} = 1$ for all λ and that $a_{\lambda, \mu} \ne 0$ implies that $\mu \le \lambda$ (dominance order). For any ordering of the partitions, the matrix $(a_{\lambda, \mu})$ is invertible (i.e., has determinant ± 1).*

The same conclusion holds if instead we assume that $a_{\lambda, \lambda^\dagger} = 1$ for all λ and that $a_{\lambda, \mu} \ne 0$ implies that $\mu \le \lambda^\dagger$.

---

[1]In the **English convention** (which is the one that we will use), row i sits above row $i + 1$, in the **French convention**, it is reversed. There is also the **Russian convention**, which is obtained from the English convention by rotating by 135 degrees counter-clockwise.

Proof. Note that if $\mu \leq \lambda$, then $\mu \leq^R \lambda$ (lexicographic order): suppose that $\lambda_1 = \mu_1, \ldots, \lambda_i = \mu_i$ but $\lambda_{i+1} \neq \mu_{i+1}$. Using the dominance order, $\lambda_1 + \cdots + \lambda_{i+1} > \mu_1 + \cdots + \mu_{i+1}$, so we conclude that $\lambda_{i+1} > \mu_{i+1}$.

Now write down the matrix $(a_{\lambda,\mu})$ so that both the rows and columns are ordered by lexicographic ordering. Then this matrix has 1's on the diagonal and is lower-triangular. In particular, its determinant is 1, so it is invertible. Any other choice of ordering amounts to conjugating by a permutation matrix, which only changes the sign of the matrix.

In the second case, write down the matrix $(a_{\lambda,\mu})$ with respect to the lexicographic ordering $\lambda(1), \lambda(2), \ldots, \lambda(p(n))$ for rows, but with respect to the ordering $\lambda(1)^\dagger, \lambda(2)^\dagger, \ldots, \lambda(p(n))^\dagger$ for columns. This matrix again has 1's on the diagonal and is upper-triangular, so has determinant 1. If we want to write down the matrix with the same ordering for both rows and columns, we just need to permute the columns which changes the determinant by a sign only. $\qquad\square$

2.2. Tabloids.

Let n be a positive integer and $\lambda = (\lambda_1, \ldots, \lambda_r)$ a partition of n. A λ-**tableau** is a filling of the boxes of the Young diagram with the numbers $1, \ldots, n$, each appearing exactly once. The symmetric group $\mathfrak{S}_n$ acts on the set of tableaux by permuting values. Given λ-tableaux t_1, t_2, we write $t_1 \sim t_2$ if t_2 is obtained from t_1 by rearranging the entries within each row. For example:

$$
\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\\hline 4 & 5 \\\cline{1-2}\end{array} \sim \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\\hline 5 & 4 \\\cline{1-2}\end{array}
$$

Then $\sim$ is an equivalence relation, and we define a λ-**tabloid** to be an equivalence class of λ-tableaux. Given a λ-tableau t, we let $\{t\}$ denote its equivalence class. This set also carries an action of $\mathfrak{S}_n$. The action is transitive and the stabilizer of a fixed λ-tabloid is isomorphic to $\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_r}$.

Lemma 2.2.1. *Let λ, μ be partitions of n. Let t_1 be a λ-tableau and t_2 be a μ-tableau. Suppose that for every i, the numbers from the ith row of t_2 belong to different columns of t_1. Then $\lambda \geq \mu$ (dominance order).*

Proof. There are μ_1 entries in the first row of t_2. If they are in different columns of t_1, then λ has at least μ_1 many columns, i.e., $\lambda_1 \geq \mu_1$. Next, there are μ_2 entries in the second row of t_2 which are also in different columns of t_1. Then delete all of the entries from t_1 that don't come from the first two rows of t_2 and shift all of the entries to the top. They must then occupy the first two rows, so that $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$. More generally, if we do this procedure for the entries in the first i rows of t_2, we conclude that $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$. In particular, $\lambda \geq \mu$. $\qquad\square$

2.3. Specht modules.

At first, we will work over an arbitrary field $\mathbf{k}$ and later assume that it is of characteristic 0 to get stronger results. Keep the notation from the previous section.

We let $\mathbf{M}^\lambda$ denote the permutation representation of $\mathfrak{S}_n$ on λ-tabloids. From the above discussion, $\mathbf{M}^\lambda \cong \mathrm{Ind}_{\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r}}^{\mathfrak{S}_n} \mathbf{k}$.

Let t be a λ-tableau. Let $C_t \subset \mathfrak{S}_n$ be the subgroup of permutations which preserve the set of entries in each column of t. Define the **signed column sum** $\kappa_t \in \mathbf{k}[\mathfrak{S}_n]$ by

$$
\kappa_t = \sum_{\sigma \in C_t} \mathrm{sgn}(\sigma)\sigma.
$$

Note that κ_t depends on the tableau and if $t' \sim t$, we do not necessarily have equality of their signed column sums. Define the **polytabloid** $e_t \in \mathbf{M}^\lambda$ by

$$e_t = \kappa_t \cdot \{t\}.$$

Example 2.3.1. Let $t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\\hline 4 & 5 \\\cline{1-2}\end{array}$. Then

$$e_t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\\hline 4 & 5 \\\cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 4 & 2 & 3 \\\hline 1 & 5 \\\cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 1 & 5 & 3 \\\hline 4 & 2 \\\cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 4 & 5 & 3 \\\hline 1 & 2 \\\cline{1-2}\end{array}. \qquad \square$$

Define $\mathbf{S}^\lambda$ to be the $\mathbf{k}$-linear span of the e_t. This is called the **Specht module**.

Example 2.3.2. If $\lambda = (n)$, then there is only one λ-tabloid, so that $\mathbf{S}^{(n)} = \mathbf{M}^{(n)} = \mathbf{k}$ is the trivial representation.

Now consider $\lambda = (n-1, 1)$. Then there are n λ-tabloids since the only relevant datum is which number goes in the second row. Let x_i be the λ-tabloid where i appears in the second row. So $\mathbf{M}^{(n-1,1)} = \mathbf{k}^n$ is the permutation representation of $\mathfrak{S}_n$. Then for a tableau with i in the second row and j in the first entry of the first row, we have $e_t = x_j - x_i$. We see that $\mathbf{S}^{(n-1,1)}$ is the subspace of $\mathbf{k}^n$ consisting of linear combinations of tabloids whose coefficients sum to 0. $\qquad \square$

Lemma 2.3.3. $\mathbf{S}^\lambda$ *is a* $\mathfrak{S}_n$*-subrepresentation of* $\mathbf{M}^\lambda$. *Furthermore, it is generated, as a* $\mathbf{k}[\mathfrak{S}_n]$*-module, by* e_t *for any tableau* t.

Proof. For any $\sigma \in \mathfrak{S}_n$ and λ-tableau t, we have $\sigma C_t \sigma^{-1} = C_{\sigma t}$ and hence $\sigma \kappa_t \sigma^{-1} = \kappa_{\sigma t}$, which means that

$$\sigma \cdot e_t = \sigma \kappa_t \{t\} = \kappa_{\sigma t} \sigma \{t\} = \kappa_{\sigma t} \{\sigma t\} = e_{\sigma t}.$$

Hence $\mathfrak{S}_n$ preserves the subspace $\mathbf{M}^\lambda$ and we see that given any e_t, we can generate all of the others using $\mathfrak{S}_n$. $\qquad \square$

Lemma 2.3.4. *Let* λ, μ *be partitions of* n. *Let* t_1 *be a* λ*-tableau and let* t_2 *be a* μ*-tableau. Suppose that* $\kappa_{t_1}\{t_2\} \neq 0$. *Then* $\lambda \geq \mu$. *Furthermore, if* $\lambda = \mu$, *then* $\kappa_{t_1}\{t_2\} = \pm e_{t_1}$.

Proof. We will use Lemma 2.2.1. Let a, b be numbers in the same row of t_2. Then $(1 - (a, b))\{t_2\} = 0$. Suppose that a, b are in the same column of t_1. Then $(a, b) \in C_{t_1}$, so pick coset representatives $\sigma_1, \ldots, \sigma_k$ for $C_{t_1}/\{1, (a, b)\}$. Then

$$\kappa_{t_1} = \sum_{i=1}^{k} \mathrm{sgn}(\sigma_i)\sigma_i \cdot (1 - (a, b)),$$

and so $\kappa_{t_1}\{t_2\} = 0$. In particular, if $\kappa_{t_1}\{t_2\} \neq 0$, then a, b must be in different columns of t_1. Using Lemma 2.2.1, we conclude that $\lambda \geq \mu$.

If we further know that $\lambda = \mu$, then we claim there is a permutation $\sigma \in C_{t_1}$ so that $\sigma\{t_2\} = \{t_1\}$. To see this, note that for every pair a, b in the same row of t_1, we have shown that they are in different columns of t_2. Hence, we can pick σ so that it moves a, b both to the corresponding row. In particular, $\kappa_{t_1}\{t_2\} = \kappa_{t_1}\sigma^{-1}\{t_1\} = \mathrm{sgn}(\sigma^{-1})\kappa_{t_1}\{t_1\} = \pm e_{t_1}$. $\qquad \square$

Corollary 2.3.5. *Pick* $u \in \mathbf{M}^\lambda$ *and a* λ*-tableau* t. *Then* $\kappa_t u$ *is a scalar multiple of* e_t.

Proof. u is a linear combination of tabloids $\{t'\}$. From the previous lemma, $\kappa_t\{t'\}$ is either 0 or $\pm e_t$, so the claim follows. $\qquad \square$

Define a symmetric bilinear form $\langle,\rangle$ on $\mathbf{M}^\lambda$ by making the tabloids an orthonormal basis:

$$\langle \{t_1\}, \{t_2\} \rangle = \begin{cases} 1 & \text{if } t_1 \sim t_2 \\ 0 & \text{else} \end{cases}$$

and extending linearly. It is immediate that this is an $\mathfrak{S}_n$-invariant form, i.e., $\langle \sigma v_1, \sigma v_2 \rangle = \langle v_1, v_2 \rangle$ for any $v_1, v_2 \in \mathbf{M}^\lambda$.

Lemma 2.3.6. *For $u, v \in \mathbf{M}^\lambda$ and a λ-tableau t, we have*

$$\langle \kappa_t u, v \rangle = \langle u, \kappa_t v \rangle.$$

Proof. Using that C_t is a subgroup and that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$, we have

$$\langle \kappa_t u, v \rangle = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma) \langle \sigma u, v \rangle = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma) \langle u, \sigma^{-1} v \rangle = \langle u, \kappa_t v \rangle. \qquad \square$$

Given a subspace $V \subseteq \mathbf{M}^\lambda$, we define the orthogonal complement $V^\perp = \{v \in \mathbf{M}^\lambda \mid \langle v, w \rangle = 0 \text{ for all } w \in V\}$.

Theorem 2.3.7 (Submodule theorem). *If $U \subseteq \mathbf{M}^\lambda$ is a subrepresentation, then either $\mathbf{S}^\lambda \subseteq U$ or $U \subseteq (\mathbf{S}^\lambda)^\perp$.*

Proof. Pick $u \in U$. By Corollary 2.3.5, $\kappa_t u$ is a scalar multiple of e_t. If there exists u and t so that this multiple is nonzero, then $e_t \in U$. Since e_t generates $\mathbf{S}^\lambda$ as a $\mathbf{k}[\mathfrak{S}_n]$-module, we conclude that $\mathbf{S}^\lambda \subseteq U$.

Otherwise, we are in the situation that $\kappa_t u = 0$ for all $u \in U$ and all λ-tableau t. Then we have

$$0 = \langle \kappa_t u, \{t\} \rangle = \langle u, \kappa_t \{t\} \rangle = \langle u, e_t \rangle$$

for all λ-tableau t. This means that $u \in (\mathbf{S}^\lambda)^\perp$ so that $U \subseteq (\mathbf{S}^\lambda)^\perp$. $\qquad \square$

Corollary 2.3.8. $\mathbf{S}^\lambda / (\mathbf{S}^\lambda \cap (\mathbf{S}^\lambda)^\perp)$ *is either 0, or absolutely irreducible, i.e., irreducible after any enlargement of the field of coefficients.*

Proof. The submodule theorem tells us that any subrepresentation of $\mathbf{S}^\lambda$ must either be all of $\mathbf{S}^\lambda$ or be contained in $\mathbf{S}^\lambda \cap (\mathbf{S}^\lambda)^\perp$. In particular, $\mathbf{S}^\lambda / (\mathbf{S}^\lambda \cap (\mathbf{S}^\lambda)^\perp)$ is either 0 or irreducible. Now we use the following fact (left to exercises): given any finite-dimensional vector space V with a basis $v_1, \ldots, v_m$ with a symmetric bilinear form $\langle,\rangle$, the dimension of $V/V^\perp$ is the rank of the Gram matrix $(\langle v_i, v_j \rangle)_{i,j=1,\ldots,m}$. In our setting, $V = \mathbf{S}^\lambda$ and $\langle,\rangle$ is the restriction of the form from $\mathbf{M}^\lambda$. In this case, we take our basis to be a maximal size linearly independent set of polytabloids. In that case, the values $\langle v_i, v_j \rangle$ are all integers (or elements in $\mathbf{Z}/p$), and in particular, the rank of the matrix is not affected by enlarging the field, and hence the intersection $\mathbf{S}^\lambda \cap (\mathbf{S}^\lambda)^\perp$ does not increase in dimension. $\qquad \square$

Theorem 2.3.9. *If $\mathbf{k}$ is a field of characteristic 0, then $\mathbf{S}^\lambda$ is an irreducible representation of $\mathfrak{S}_n$.*

Proof. It suffices to prove this when $\mathbf{k} = \mathbf{Q}$ from the previous corollary. In that case, $\langle,\rangle$ is an inner product, i.e., $\langle u, u \rangle > 0$ whenever $u \neq 0$. This means that $U \cap U^\perp = 0$ for any subspace U of $\mathbf{M}^\lambda$, and in particular, $\mathbf{S}^\lambda / (\mathbf{S}^\lambda \cap (\mathbf{S}^\lambda)^\perp) = \mathbf{S}^\lambda$ is either 0 or irreducible. But it is not 0 since each e_t is a nonzero element. $\qquad \square$

Hence have associated with each partition λ of n an irreducible representation $\mathbf{S}^\lambda$ of $\mathfrak{S}_n$. Furthermore, all of these representations can be constructed over $\mathbf{Q}$. From our previous discussion, we know that the number of partitions of n is the same as the number of conjugacy classes of $\mathfrak{S}_n$. So if we can show that $\mathbf{S}^\lambda \not\cong \mathbf{S}^\mu$ whenever $\lambda \neq \mu$, then we have constructed all of the isomorphism types of irreducible representations of $\mathfrak{S}_n$.

Lemma 2.3.10. *Let $\mathbf{k}$ be a field of characteristic 0. Suppose that there is a nonzero $\mathfrak{S}_n$-equivariant homomorphism $\theta\colon \mathbf{S}^\lambda \to \mathbf{M}^\mu$. Then $\lambda \geq \mu$.*

Proof. Let t be a λ-tableau. Then $\theta(e_t) \neq 0$ since $\mathbf{S}^\lambda$ is generated under the $\mathfrak{S}_n$-action by e_t and $\theta \neq 0$. In particular, $\theta(e_t) = \theta(\kappa_t\{t\}) = \kappa_t \theta(\{t\})$. The last term is a sum of elements of the form $\kappa_t\{t'\}$ where t' is a μ-tableau, so at least one of them must be nonzero, and so by Lemma 2.3.4, $\lambda \geq \mu$. $\qquad\square$

Corollary 2.3.11. *Let $\mathbf{k}$ be a field of characteristic 0. If $\lambda \neq \mu$, then $\mathbf{S}^\lambda \not\cong \mathbf{S}^\mu$.*

Proof. Suppose that $\mathbf{S}^\lambda \cong \mathbf{S}^\mu$. Then we have a nonzero homomorphism $\theta\colon \mathbf{S}^\lambda \to \mathbf{M}^\mu$ by composing the isomorphism with the inclusion $\mathbf{S}^\mu \subseteq \mathbf{M}^\mu$. Then Lemma 2.3.10 tells us that $\lambda \geq \mu$. On the other hand, we also have a nonzero homomorphism $\mathbf{S}^\mu \to \mathbf{M}^\lambda$ by using the inverse isomorphism, and so $\mu \geq \lambda$. This implies that $\lambda = \mu$. $\qquad\square$

Hence we have given a construction for all of the irreducible representations of $\mathfrak{S}_n$ up to isomorphism. But there are many basic things we cannot easily deduce from this construction, for example, what is the dimension of $\mathbf{S}^\lambda$. Since the polytabloids e_t span $\mathbf{S}^\lambda$, it suffices to find a maximal set of them which is linearly independent. We describe how to do this next.

2.4. **Garnir relations and standard tableaux.** As we have seen, the polytabloids e_t are linearly dependent in general. Our goal is to systematically construct linear dependencies which allow us to conclude that a particular subset of e_t form a basis for $\mathbf{S}^\lambda$.

Let λ be a partition and let t be a λ-tableau. Let X be a subset of values of the boxes in the ith column of t and let Y be a subset of values of the boxes in the $(i+1)$th column of t. Let $\mathfrak{S}_X$ denote the group of permutations that permute the elements of X but leave all other elements fixed, and similarly define $\mathfrak{S}_Y$ and $\mathfrak{S}_{X \cup Y}$. Pick coset representatives $\sigma_1, \ldots, \sigma_k$ for $\mathfrak{S}_{X \cup Y}/(\mathfrak{S}_X \times \mathfrak{S}_Y)$.

We define the **Garnir element** to be

$$G_{X,Y} = \sum_{j=1}^k \operatorname{sgn}(\sigma_j)\sigma_j.$$

Example 2.4.1. Let $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$, $X = \{3,5\}$, $Y = \{2,4\}$. To define coset representatives for $\mathfrak{S}_{X \cup Y}/\mathfrak{S}_X \times \mathfrak{S}_Y$, we write them as $w(3)w(5)w(2)w(4)$, and we can assume that $w(2) < w(4)$ and $w(3) < w(5)$. The representatives we get are

$$2345, 2435, 2534, 3425, 3524, 4523$$

and

$$G_{X,Y}e_t = -e_{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}} + e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}} - e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}} - e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}} + e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}} - e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline 5 \\ \cline{1-1}\end{array}}. \qquad\square$$

Theorem 2.4.2 (Garnir relations). *If $|X \cup Y| > \lambda_i^\dagger$, then $G_{X,Y} e_t = 0$.*

Proof. The identity only involves integer coefficients, so it will suffice to prove that it holds when $\mathbf{k} = \mathbf{Q}$ (we can reduce the coefficients modulo p to see its validity in $\mathbf{F}_p$). Define

$$\alpha = \sum_{\sigma \in \mathfrak{S}_X \times \mathfrak{S}_Y} \mathrm{sgn}(\sigma)\sigma, \qquad \beta = \sum_{\sigma \in \mathfrak{S}_{X \cup Y}} \mathrm{sgn}(\sigma)\sigma.$$

Recall that C_t is the subgroup of $\mathfrak{S}_n$ that preserves the columns of t. Since $|X \cup Y| > \lambda_i^\dagger$, for every $\tau \in C_t$, there are always two values from the boxes of $X \cup Y$ that are in the same row of τt. In particular, we have $\beta\{\tau t\} = 0$ (let ρ be the transposition swapping these two elements, then $\sigma \mapsto \sigma\rho$ gives a sign-reversing bijection on $\mathfrak{S}_{X \cup Y}$ such that $\sigma\{\tau t\} = \sigma\rho\{\tau t\}$). In particular, $\beta e_t = \beta\kappa_t\{t\} = 0$.

Next, α is a factor of κ_t. To be precise, $\mathfrak{S}_X \times \mathfrak{S}_Y$ is a subgroup of C_t, so if we pick coset representatives, then their signed sum multiplied by α gives κ_t. Similarly, $\beta = G_{X,Y}\alpha$. Also, for any $\sigma \in \mathfrak{S}_X \times \mathfrak{S}_Y$, we have $\mathrm{sgn}(\sigma)\sigma\kappa_t = \kappa_t$ since multiplication by σ permutes the elements of C_t and hence does not affect the sum. So

$$0 = \beta\kappa_t\{t\} = |X|!|Y|!G_{X,Y}\kappa_t\{t\}.$$

Since $\mathbf{k} = \mathbf{Q}$, we can divide by $|X|!|Y|!$ to see that $G_{X,Y}e_t = 0$. $\qquad\square$

A tableau t is **standard** if the numbers increase left to right in each row, and top to bottom in each column. In that case, we say that e_t is a **standard polytabloid** and that $\{t\}$ is a **standard tabloid**. Our goal now is to show that the standard polytabloids form a basis for $\mathbf{S}^\lambda$.

We define a total ordering on λ-tabloids as follows: $\{t_1\} < \{t_2\}$ if for some i:

(1) When $j > i$, j is in the same row of $\{t_1\}$ and $\{t_2\}$,
(2) i is in a higher row of $\{t_1\}$ than in $\{t_2\}$.

In our proof, we will also make use of a column-analogue of tabloids. Namely, if t is a tableau, then $[t]$ denotes the set of all tableaux which have the same values in each column as t (though possibly in a different order). We define an ordering $[t_1] < [t_2]$ just as above with "row" replaced by "column".

Theorem 2.4.3. *The set of standard λ-polytabloids is a basis for $\mathbf{S}^\lambda$.*

Proof. First we establish linear independence. Let t be a standard tableau. Then for any $\sigma \in C_t$, we have $\{t\} > \{\sigma t\}$. Let $t_1, \ldots, t_r$ be distinct standard tableaux such that $\{t_1\} < \cdots < \{t_r\}$ and suppose $c_1 e_{t_1} + \cdots + c_r e_{t_r} = 0$. Then $\{t_r\}$ is the largest tabloid appearing in any of these terms, so $c_r = 0$. By induction, we deduce that in fact all $c_i = 0$.

Next, we show that every polytabloid e_t can be written as a linear combination of standard ones. Note that it suffices to do this for one e_t per column equivalence class since $\sigma e_t = \mathrm{sgn}(\sigma)e_t$ whenever $\sigma \in C_t$. So we will prove the statement by descending induction on the total ordering of column equivalence classes. The largest column equivalence class is obtained by putting the numbers $1, \ldots, n$ by first filling up the first column in order with $1, \ldots, \lambda_1^\dagger$, then the next $\lambda_2^\dagger$ entries in order in the next column, etc. This is a standard tableau, so there is nothing to prove in the base case.

Now let t be a general non-standard tableau. By induction, we assume that for all t' such that $[t'] > [t]$, we have $e_{t'}$ is a linear combination of standard polytabloids. We may also assume that t is increasing going top to bottom in each column since $\sigma e_t = \mathrm{sgn}(\sigma)e_t$ for any

$\sigma \in C_t$. Since t is not standard, for some i and j, we have $t_{j,i} > t_{j,i+1}$. Let X be the set of values in the ith column of t in rows j and below, and let Y be the set of values in the $(i+1)$th column of t in rows j and above. Pick coset representatives $\sigma_1, \ldots, \sigma_k$ to define the Garnir element $G_{X,Y}$, and choose them so that σ_1 is the identity element representing the identity coset. Then by Theorem 2.4.2, we have $G_{X,Y}e_t = 0$, i.e., that

$$e_t = -\sum_{r=1}^{k} \mathrm{sgn}(\sigma_r) e_{\sigma_r t}.$$

By our choice of X, Y, we have

$$t_{1,i+1} < t_{2,i+1} < \cdots < t_{j,i+1} < t_{j,i} < t_{j+1,i} < \cdots < t_{\lambda_i^\dagger, i}.$$

Hence each non-identity σ_r is simply moving smaller numbers to later columns, which means that $[\sigma_r t] > [t]$ so each $e_{\sigma_r t}$ is a linear combination of standard polytabloids. By induction, we conclude that e_t is a linear combination of standard polytabloids. $\square$

Corollary 2.4.4. *The dimension of $\mathbf{S}^\lambda$ is the number of standard tableaux of shape λ. In particular, it does not depend on the field $\mathbf{k}$.*

We will later see how to get useful formulas for the number of standard tableaux. The next question is how to compute the characters of the Specht modules $\mathbf{S}^\lambda$. It turns out it is much better to do this via an indirect method. We will use this as excuse to discuss symmetric functions.

3. Symmetric functions

3.1. **Definitions.** Let $x_1, \ldots, x_n$ be a finite set of indeterminates. The symmetric group $\mathfrak{S}_n$ acts on $\mathbf{Z}[x_1, \ldots, x_n]$, the ring of polynomials in n variables and integer coefficients, by substitution of variables:

$$\sigma \cdot f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

The **ring of symmetric polynomials** is the set of fixed polynomials:

$$\Lambda(n) := \{f \in \mathbf{Z}[x_1, \ldots, x_n] \mid \sigma \cdot f = f \text{ for all } \sigma \in \mathfrak{S}_n\}.$$

This is a subring of $\mathbf{Z}[x_1, \ldots, x_n]$.

We will also treat the case $n = \infty$. Let $x_1, x_2, \ldots$ be a countably infinite set of indeterminates. Let $\mathfrak{S}_\infty$ be the group of all permutations of $\{1, 2, \ldots\}$. Consider the ring R of power series in $x_1, x_2, \ldots$ of *bounded degree* (this notation R is just for this discussion and we will not refer to it again later). Hence, elements of R can be infinite sums, but only in a finite number of degrees. Then $\mathfrak{S}_\infty$ acts on R, and we define the **ring of symmetric functions**

$$\Lambda := \{f \in R \mid \sigma \cdot f = f \text{ for all } \sigma \in \mathfrak{S}_\infty\}.$$

Again, this is a subring of R. Write $\pi_n \colon \Lambda \to \Lambda(n)$ for the homomorphism which sets $x_{n+1} = x_{n+2} = \cdots = 0$.

Remark 3.1.1. (For those familiar with inverse limits.) There is a ring homomorphism $\pi_{n+1,n} \colon \Lambda(n+1) \to \Lambda(n)$ obtained by setting $x_{n+1} = 0$. Furthermore, $\Lambda(n) = \bigoplus_{d \geq 0} \Lambda(n)_d$ where $\Lambda(n)_d$ is the subgroup of homogeneous symmetric polynomials of degree d. The map $\pi_{n+1,n}$ restricts to a map $\Lambda(n+1)_d \to \Lambda(n)_d$; set

$$\Lambda_d = \varprojlim_n \Lambda(n)_d.$$

Then
$$\Lambda = \bigoplus_{d \geq 0} \Lambda_d.$$

Note that we aren't saying that Λ is the inverse limit of the $\Lambda(n)$ as rings; the latter object includes infinite sums of unbounded degree. The correct way to say this is that Λ is the inverse limit of the $\Lambda(n)$ as *graded* rings. $\square$

Example 3.1.2. Here are some basic examples of elements in Λ (we will study them more soon):

$$p_k := \sum_{i \geq 1} x_i^k$$

$$e_k := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

$$h_k := \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}. \qquad \square$$

Sometimes, we want to work with rational coefficients instead of integer coefficients. In that case, we'll write $\Lambda_{\mathbf{Q}}$ or $\Lambda(n)_{\mathbf{Q}}$ to denote the appropriate rings.

3.2. **Monomial symmetric functions.** Given an infinite sequence $(\alpha_1, \alpha_2, \dots)$ with finitely many nonzero entries, we use x^α as a convention for $\prod_{i \geq 1} x_i^{\alpha_i}$. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, define the **monomial symmetric function** by

$$m_\lambda = \sum_\alpha x^\alpha$$

where the sum is over all *distinct* permutations α of λ. This is symmetric by definition. So for example, $m_1 = \sum_{i \geq 1} x_i$ since all of the distinct permutations of $(1, 0, 0, \dots)$ are integer sequences with a single 1 somewhere and 0 elsewhere. By convention, $m_0 = 1$. Some other examples:

$$m_{1,1} = \sum_{i < j} x_i x_j$$

$$m_{3,2,1} = \sum_{\substack{i,j,k \\ i \neq j,\ j \neq k,\ i \neq k}} x_i^3 x_j^2 x_k.$$

In general, $m_{1^k} = e_k$ and $m_k = p_k$.

Theorem 3.2.1. *As we range over all partitions, the m_λ form a basis for Λ.*

Proof. They are linearly independent since no two m_λ have any monomials in common. Clearly they also span: given $f \in \Lambda$, we can write $f = \sum_\alpha c_\alpha x^\alpha$ and $c_\alpha = c_\beta$ if both are permutations of each other, so this can be rewritten as $f = \sum_\lambda c_\lambda m_\lambda$ where the sum is now over just the partitions. $\square$

Corollary 3.2.2. Λ_d *has a basis given by* $\{m_\lambda \mid |\lambda| = d\}$, *and hence is a free abelian group of rank* $p(d) = |\mathrm{Par}(d)|$.

Theorem 3.2.3. $\Lambda(n)_d$ *has a basis given by* $\{m_\lambda(x_1, \dots, x_n) \mid |\lambda| = d,\ \ell(\lambda) \leq n\}$.

3.3. **Elementary symmetric functions.** Recall that we defined

$$e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, define the **elementary symmetric function** by

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}.$$

Note $e_\lambda \in \Lambda_{|\lambda|}$.

Since the m_μ form a basis for Λ (Theorem 3.2.1), we have expressions

$$e_\lambda = \sum_\mu M_{\lambda,\mu} m_\mu.$$

We can give an interpretation for these change of basis coefficients as follows. Given an (infinite) matrix A with finitely many nonzero entries, let $\mathrm{row}(A) = (\sum_{i \geq 1} A_{1,i}, \sum_{i \geq 1} A_{2,i}, \ldots)$ be the sequence of row sums of A, and let $\mathrm{col}(A)$ be the sequence of column sums of A. A $(0,1)$-matrix is one whose entries are only 0 or 1.

Lemma 3.3.1. $M_{\lambda,\mu}$ *is the number of* $(0,1)$*-matrices* A *with* $\mathrm{row}(A) = \lambda$ *and* $\mathrm{col}(A) = \mu$.

Proof. To get a monomial in e_λ, we have to choose monomials from each e_{λ_i} to multiply. Each monomial of e_{λ_i} can be represented by a subset of $\{1, 2, \ldots\}$ of size λ_i, or alternatively, as a sequence (thought of as a row vector) of 0's and 1's where a 1 is put in each place of the subset. Hence, we can represent each choice of multiplying out a monomial by concatenating these row vectors to get a matrix A. By definition, $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$ where the monomial we get is x^μ. $\square$

Corollary 3.3.2. $M_{\lambda,\mu} = M_{\mu,\lambda}$.

Proof. Take the transpose of each $(0,1)$-matrix to get the desired bijection. $\square$

Theorem 3.3.3. *If* $M_{\lambda,\mu} \neq 0$, *then* $\mu \leq \lambda^\dagger$. *Furthermore,* $M_{\lambda,\lambda^\dagger} = 1$. *In particular, the* e_λ *form a basis of* Λ.

Proof. Suppose $M_{\lambda,\mu} \neq 0$. Then there is a $(0,1)$-matrix A with $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$. Now let A' be obtained from A by left-justifying all of the 1's in each row (i.e., move all of the 1's in row i to the first λ_i positions). Note that $\mathrm{col}(A') = \lambda^\dagger$. Also, the number of 1's in the first i columns of A' is at least as many as the number of 1's in the first i columns of A, so $\lambda_1^\dagger + \cdots + \lambda_i^\dagger \geq \mu_1 + \cdots + \mu_i$, i.e., $\lambda^\dagger \geq \mu$. Moreover, if $\mu = \lambda^\dagger$, A' is the only $(0,1)$-matrix with $\mathrm{row}(A') = \lambda$ and $\mathrm{col}(A') = \lambda^\dagger$.

The second statement follows from Lemma 2.1.2. $\square$

Theorem 3.3.4. *The set* $\{e_\lambda(x_1, \ldots, x_n) \mid \lambda_1 \leq n,\ |\lambda| = d\}$ *is a basis of* $\Lambda(n)_d$.

Proof. If $\lambda_1 > n$, then $e_{\lambda_1}(x_1, \ldots, x_n) = 0$, so $e_\lambda(x_1, \ldots, x_n) = 0$. Hence under the map $\pi_n \colon \Lambda \to \Lambda(n)$, the proposed e_λ span the image. The number of such e_λ in degree d is $|\{\lambda \mid \lambda_1 \leq n,\ |\lambda| = d\}|$, which is the same as $|\{\lambda \mid \ell(\lambda) \leq n,\ |\lambda| = d\}|$ via the transpose $\dagger$, and this is the rank of $\Lambda(n)_d$, so the e_λ form a basis. $\square$

Remark 3.3.5. The previous two theorems say that the elements $e_1, e_2, e_3, \ldots$ are algebraically independent in Λ, and that the elements $e_1, \ldots, e_n$ are algebraically independent in $\Lambda(n)$. This is also known as the "fundamental theorem of symmetric functions". $\square$

3.4. **The involution** ω. Since the e_i are algebraically independent, we can define a ring homomorphism $f\colon \Lambda \to \Lambda$ by specifying $f(e_i)$ arbitrarily.[2] Define

$$\omega\colon \Lambda \to \Lambda$$

by $\omega(e_i) = h_i$, where recall that $h_k = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}$.

Theorem 3.4.1. ω *is an involution, i.e., $\omega^2 = 1$. Equivalently, $\omega(h_i) = e_i$.*

Proof. Consider the ring $\Lambda[\![t]\!]$ of power series in t with coefficients in Λ. Define two elements of $\Lambda[\![t]\!]$:

$$(3.4.2) \qquad\qquad E(t) = \sum_{n \geq 0} e_n t^n, \qquad H(t) = \sum_{n \geq 0} h_n t^n.$$

Note that $E(t) = \prod_{i \geq 1}(1 + x_i t)$ (by convention, the infinite product means we have to choose 1 all but finitely many times; if you multiply it out, the coefficient of t^n is all ways of getting a monomial $x_{i_1} \cdots x_{i_n}$ with $i_1 < \cdots < i_n$ and each one appears once, so it is e_n) and that $H(t) = \prod_{i \geq 1}(1 - x_i t)^{-1}$ (same as for $E(t)$ but use the geometric sum $(1 - x_i t)^{-1} = 1 + \sum_{d > 0} x_i^d t^d$).

This implies that $E(t)H(-t) = 1$. The coefficient of t^n on the left side of this identity is $\sum_{i=0}^{n}(-1)^{n-i} e_i h_{n-i}$. In particular, that sum is 0 for $n > 0$. Now apply ω to that sum and multiply by $(-1)^n$ to get

$$\sum_{i=0}^{n}(-1)^i h_i \omega(h_{n-i}) = 0.$$

This shows that $\sum_{n \geq 0} \omega(h_n) t^n = H(-t)^{-1} = E(t)$, so $\omega(h_n) = e_n$. $\qquad\square$

Furthermore, we can define a finite analogue of ω, the ring homomorphism $\omega_n\colon \Lambda(n) \to \Lambda(n)$, given by $\omega_n(e_i) = h_i$ for $i = 1, \ldots, n$.

Theorem 3.4.3. $\omega_n^2 = 1$, *and ω_n is invertible. Equivalently, $\omega_n(h_i) = e_i$ for $i = 1, \ldots, n$.*

Proof. In Λ, we have expressions $h_i = \sum_{|\lambda|=i} c_{i,\lambda} e_\lambda$. We know that $\omega(h_i) = e_i$, so we also get $e_i = \sum_{|\lambda|=i} c_{i,\lambda} \omega(e_\lambda)$.

Using the first relation, we also get $\omega_n(\pi_n(h_i)) = \sum_{|\lambda|=i} c_{i,\lambda} \omega_n(\pi_n(e_\lambda))$. By definition, $\omega_n(\pi_n(e_\lambda)) = \pi_n(\omega(e_\lambda))$ when $\lambda_1 \leq n$. This condition is guaranteed if $i \leq n$, so we can rewrite it as $\omega_n(\pi_n(h_i)) = \sum_{|\lambda|=i} c_{i,\lambda} \pi_n(\omega(e_\lambda))$. Finally, applying π_n to the second relation above, we get $\pi_n(e_i) = \sum_{|\lambda|=i} c_{i,\lambda} \pi_n(\omega(e_\lambda))$, so we conclude that $\pi_n(e_i) = \omega_n(\pi_n(h_i))$, as desired. $\qquad\square$

3.5. **Complete homogeneous symmetric functions.** For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, define the **complete homogeneous symmetric functions** by

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}.$$

Theorem 3.5.1. *The h_λ form a basis for Λ.*

Proof. Since ω is a ring homomorphism, $\omega(e_\lambda) = h_\lambda$. Now use the fact that the e_λ form a basis (Theorem 3.3.3) and that ω is an isomorphism (Theorem 3.4.1). $\qquad\square$

---

[2]Every element is uniquely of the form $\sum_\lambda c_\lambda e_\lambda$; since f is a ring homomorphism, it sends this to $\sum_\lambda c_\lambda f(e_{\lambda_1}) f(e_{\lambda_2}) \cdots f(e_{\lambda_{\ell(\lambda)}})$.

Again, we can write h_λ in terms of m_μ:

$$h_\lambda = \sum_\mu N_{\lambda,\mu} m_\mu$$

and give an interpretation for the coefficients. This is similar to $M_{\lambda,\mu}$: the $N_{\lambda,\mu}$ is the number of matrices A with non-negative integer entries such that $\mathrm{row}(A) = \lambda$ and $\mathrm{col}(A) = \mu$ (not just $(0,1)$-matrices). The proof is similar to the $M_{\lambda,\mu}$ case. However, it does not satisfy any upper-triangularity properties, so it is not as easy to see directly (without using ω) that the h_λ are linearly independent.

Theorem 3.5.2. $h_1, \ldots, h_n$ *are algebraically independent generators of* $\Lambda(n)$, *and the set* $\{h_\lambda(x_1, \ldots, x_n) \mid \lambda_1 \le n, \ |\lambda| = d\}$ *is a basis of* $\Lambda(n)_d$.

Proof. Follows from Theorem 3.3.4 and Theorem 3.4.3. $\qquad\square$

3.6. **Power sum symmetric functions.** Recall we defined

$$p_k = \sum_{n \ge 1} x_n^k.$$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ (here we assume $\lambda_k > 0$), the **power sum symmetric functions** are defined by

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}.$$

We can write the p_λ in terms of the m_μ:

$$p_\lambda = \sum_\mu R_{\lambda,\mu} m_\mu.$$

Recall the definitions of $E(t)$ and $H(t)$ from (3.4.2):

$$E(t) = \sum_{n \ge 0} e_n t^n, \qquad H(t) = \sum_{n \ge 0} h_n t^n.$$

Define $P(t) \in \Lambda[\![t]\!]$ by

$$P(t) = \sum_{n \ge 1} p_n t^{n-1}$$

(note the unconventional indexing).

Lemma 3.6.1. *We have the following identities:*

$$P(t) = \frac{d}{dt} \log H(t), \qquad P(-t) = \frac{d}{dt} \log E(t).$$

Proof. We have

$$P(t) = \sum_{n \geq 1} \sum_{i \geq 1} x_i^n t^{n-1}$$

$$= \sum_{i \geq 1} \frac{x_i}{1 - x_i t}$$

$$= \sum_{i \geq 1} \frac{d}{dt} \log\left(\frac{1}{1 - x_i t}\right)$$

$$= \frac{d}{dt} \log\left(\prod_{i \geq 1} \frac{1}{1 - x_i t}\right)$$

$$= \frac{d}{dt} \log H(t).$$

The other identity is similar. $\qquad\square$

Given a partition λ, recall that $m_i(\lambda)$ is the number of times that i appears in λ. Define

$$(3.6.2) \qquad z_\lambda := \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!, \qquad \varepsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)}.$$

Theorem 3.6.3. *We have the following identities in $\Lambda[\![t]\!]$:*

$$E(t) = \sum_\lambda \varepsilon_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|}, \qquad H(t) = \sum_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|},$$

$$e_n = \sum_{|\lambda|=n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda, \qquad h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda.$$

Proof. From Lemma 3.6.1, we have $P(t) = \frac{d}{dt} \log H(t)$. Integrate both sides (and get the boundary conditions right using that $\log H(0) = 0$) and apply the exponential map:

$$H(t) = \exp\left(\sum_{n \geq 1} \frac{p_n t^n}{n}\right)$$

$$= \prod_{n \geq 1} \exp\left(\frac{p_n t^n}{n}\right)$$

$$= \prod_{n \geq 1} \sum_{d \geq 0} \frac{p_n^d t^{nd}}{n^d d!}$$

$$= \sum_\lambda \frac{p_\lambda t^{|\lambda|}}{z_\lambda}.$$

The identity for $E(t)$ is similar. Finally, the second row of identities comes from equating the coefficient of t^n in the first row of identities. $\qquad\square$

Theorem 3.6.4. *$p_1, p_2, \ldots$ are algebraically independent generators of $\Lambda_{\mathbf{Q}}$.*
$p_1(x_1, \ldots, x_n), \ldots, p_n(x_1, \ldots, x_n)$ are algebraically independent generators of $\Lambda(n)_{\mathbf{Q}}$ and the set $\{p_\lambda(x_1, \ldots, x_n) \mid \lambda_1 \leq n, |\lambda| = d\}$ is a basis for $\Lambda(n)_{\mathbf{Q},d}$.

Proof. We just explain the finite variable case, the other being similar. By Theorem 3.6.3, we have $e_i(x_1, \ldots, x_n) = \sum_{|\lambda|=i} \varepsilon_\lambda z_\lambda^{-1} p_\lambda(x_1, \ldots, x_n)$ for all i. If $i \leq n$, then this shows that $p_1, \ldots, p_n$ are algebra generators for $\Lambda(n)_\mathbf{Q}$ since the $e_1, \ldots, e_n$ are algebra generators. The space of possible monomials in the p_i of degree d is the number of λ with $\lambda_1 \leq n$ and $|\lambda| = d$, which is $\dim_\mathbf{Q} \Lambda(n)_{\mathbf{Q},d}$, so there are no algebraic relations among them. $\qquad\square$

The proof of Corollary 3.6.6 can be adapted to show that $\omega_n(p_\lambda) = \varepsilon_\lambda p_\lambda$ whenever $\lambda_1 \leq n$.

Remark 3.6.5. The p_λ do not form a basis for Λ. For example, in degree 2, we have

$$p_2 = m_2$$
$$p_{1,1} = m_2 + 2m_{1,1}$$

and the change of basis matrix has determinant 2, so is not invertible over $\mathbf{Z}$. However, they do form a basis for $\Lambda_\mathbf{Q}$. $\qquad\square$

Corollary 3.6.6. $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$, *i.e., the p_λ are a complete set of eigenvectors for ω.*

Proof. We prove this by induction on λ_1. When $\lambda_1 = 1$, this is clear since $p_{1^n} = p_1^n = e_1^n = h_1^n$ and $\varepsilon_{1^n} = 1$. So suppose we know that $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$ whenever $\lambda_1 < n$. Apply ω to the identity

$$e_n = \sum_{|\lambda|=n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda,$$

to get

$$h_n = \sum_{|\lambda|=n} \varepsilon_\lambda z_\lambda^{-1} \omega(p_\lambda).$$

Every partition satisfies $\lambda_1 < n$ except for $\lambda = (n)$, so this can be simplified to

$$h_n = \varepsilon_n z_n^{-1} \omega(p_n) + \sum_{|\lambda|=n,\ \lambda_1<n} z_\lambda^{-1} p_\lambda.$$

Compare this to the identity

$$h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda$$

to conclude that $\varepsilon_n z_n^{-1} \omega(p_n) = z_n^{-1} p_n$; now multiply both sides by z_n. Now given any other partition with $\lambda_1 = n$, use the fact that ω is a ring homomorphism, that $\varepsilon_\lambda = \varepsilon_{\lambda_1} \cdots \varepsilon_{\lambda_k}$, and that $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$ to conclude that $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$. $\qquad\square$

3.7. A scalar product. Define a bilinear form $\langle, \rangle \colon \Lambda \otimes \Lambda \to \mathbf{Z}$ by setting

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$$

where δ is the Kronecker delta (1 if $\lambda = \mu$ and 0 otherwise). In other words, if $f = \sum_\lambda a_\lambda m_\lambda$ and $g = \sum_\mu b_\mu h_\mu$, then $\langle f, g \rangle = \sum_\lambda a_\lambda b_\lambda$ (well-defined since both m and h are bases). At this point, the definition looks completely unmotivated. However, this inner product is natural from the representation-theoretic perspective, which we'll discuss later.

In our setup, m and h are dual bases with respect to the pairing. We will want a general criteria for two bases to be dual to each other. To state this criterion, we need to work in two sets of variables x and y and in the ring $\mathbf{Z}[\![x_1, x_2, \ldots, y_1, y_2, \ldots]\!]$ where the x's and y's are separately symmetric.

Lemma 3.7.1. *Let u_λ and v_μ be homogeneous bases of Λ (or $\Lambda_{\mathbf{Q}}$). Then $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda,\mu}$ if and only if*

$$\sum_\lambda u_\lambda(x)v_\lambda(y) = \prod_{i,j}(1 - x_iy_j)^{-1}.$$

Proof. Write $u_\lambda = \sum_\alpha a_{\lambda,\alpha}m_\alpha$ and $v_\mu = \sum_\beta b_{\mu,\beta}h_\beta$. Pick an ordering of the partitions of a fixed size. Write $A = (a_{\lambda,\alpha})$ and $B = (b_{\mu,\beta})$ in these orderings. First,

$$\langle u_\lambda, v_\mu \rangle = \sum_\gamma a_{\lambda,\gamma}b_{\mu,\gamma}.$$

Hence u_λ and v_μ are dual bases if and only if $\sum_\gamma a_{\lambda,\gamma}b_{\mu,\gamma} = \delta_{\lambda,\mu}$, or equivalently, $AB^T = I$ where T denotes transpose and I is the identity matrix. So A and B^T are inverses of each other, and so this is equivalent to $B^TA = I$, or $\sum_\gamma a_{\gamma,\lambda}b_{\gamma,\mu} = \delta_{\lambda,\mu}$. Finally, we have

$$\sum_\lambda u_\lambda(x)v_\lambda(y) = \sum_\lambda \sum_{\alpha,\beta} a_{\lambda,\alpha}b_{\lambda,\beta}m_\alpha(x)h_\beta(y) = \sum_{\alpha,\beta}\left(\sum_\lambda a_{\lambda,\alpha}b_{\lambda,\beta}\right)m_\alpha(x)h_\beta(y).$$

Since the $m_\alpha(x)h_\beta(y)$ are linearly independent, we see that $\sum_\gamma a_{\gamma,\lambda}b_{\gamma,\mu} = \delta_{\lambda,\mu}$ is equivalent to $\sum_\lambda u_\lambda(x)v_\lambda(y) = \sum_\lambda m_\lambda(x)h_\lambda(y)$. Now use Proposition 3.7.2 below. $\square$

Proposition 3.7.2.

$$\sum_\lambda m_\lambda(x)h_\lambda(y) = \prod_{i,j}(1 - x_iy_j)^{-1}.$$

Proof. We have

$$\prod_i\prod_j(1 - x_iy_j)^{-1} = \prod_j\sum_{n\geq 0}h_n(y)x_j^n = \sum_\alpha h_\alpha(y)x^\alpha$$

where the sum is over all sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ with finitely many nonzero entries, and $h_\alpha(y) = h_{\alpha_1}(y)h_{\alpha_2}(y)\cdots$; finally, grouping together terms α in the same $\mathfrak{S}_\infty$-orbit, the latter sum simplifies to $\sum_\lambda m_\lambda(x)h_\lambda(y)$, where the sum is now over all partitions λ. $\square$

Corollary 3.7.3. *The pairing is symmetric, i.e., $\langle f, g \rangle = \langle g, f \rangle$.*

Proof. The condition above is the same if we interchange x and y, so $\langle m_\lambda, h_\mu \rangle = \langle h_\mu, m_\lambda \rangle$. Now use bilinearity. $\square$

Proposition 3.7.4. *We have*

$$\sum_\lambda z_\lambda^{-1}p_\lambda(x)p_\lambda(y) = \prod_{i,j}(1 - x_iy_j)^{-1}.$$

In particular, $\langle p_\lambda, p_\mu \rangle = z_\lambda\delta_{\lambda,\mu}$, and p_λ is an orthogonal basis of $\Lambda_{\mathbf{Q}}$.

Proof. As in the first part of the proof of Theorem 3.6.3, we can write

$$\prod_j \prod_i (1 - x_i y_j)^{-1} = \prod_j \sum_{n \geq 0} h_n(x) y_j^n$$

$$= \prod_j \exp\left(\sum_{n \geq 1} \frac{p_n(x) y_j^n}{n}\right)$$

$$= \exp\left(\sum_{n \geq 1} \frac{p_n(x) p_n(y)}{n}\right)$$

$$= \prod_{n \geq 1} \sum_{d \geq 0} \frac{p_n(x)^d p_n(y)^d}{d! n^d}$$

$$= \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y)$$

where the final sum is over all partitions. $\square$

Corollary 3.7.5. *ω is an isometry, i.e., $\langle f, g \rangle = \langle \omega(f), \omega(g) \rangle$.*

Proof. By bilinearity, it suffices to show that this holds for any basis of $\Lambda_{\mathbf{Q}}$. We pick p_λ. Then

$$\langle \omega(p_\lambda), \omega(p_\mu) \rangle = \varepsilon_\lambda \varepsilon_\mu \langle p_\lambda, p_\mu \rangle = \varepsilon_\lambda \varepsilon_\mu z_\lambda \delta_{\lambda,\mu}$$

by Corollary 3.6.6 and the previous result. The last expression is the same as $z_\lambda \delta_{\lambda,\mu}$ since $\varepsilon_\lambda^2 = 1$, so we see that $\langle \omega(p_\lambda), \omega(p_\mu) \rangle = \langle p_\lambda, p_\mu \rangle$ for all λ, μ. $\square$

Corollary 3.7.6. *The bilinear form $\langle, \rangle$ is positive definite, i.e, $\langle f, f \rangle > 0$ for $f \neq 0$.*

Proof. Assume $f \neq 0$. Write $f = \sum_\lambda a_\lambda p_\lambda$ with some $a_\lambda \neq 0$. Then

$$\langle f, f \rangle = \sum_{\lambda, \mu} a_\lambda a_\mu \langle p_\lambda, p_\mu \rangle = \sum_\lambda z_\lambda a_\lambda^2.$$

Since $z_\lambda > 0$ and $a_\lambda^2 > 0$, we get the result. $\square$

4. Schur functions and the RSK algorithm

The goal of this section is to give several different definitions of Schur functions. They are central in many representation theoretic studies, which we will discuss later.

4.1. **Semistandard Young tableaux.** Let λ be a partition. A **semistandard Young tableau (SSYT)** T is an assignment of positive integers to the Young diagram of λ so that the numbers are weakly increasing going left to right in each row, and the numbers are strictly increasing going top to bottom in each column.

Example 4.1.1. If $\lambda = (4, 3, 1)$, and we have the assignment

a	b	c	d
e	f	g	
h			

then, in order for this to be a SSYT, we need to have

- $a \leq b \leq c \leq d$,
- $e \leq f \leq g$,

- $a < e < h$,
- $b < f$, and
- $c < g$.

An example of a SSYT is
$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 5 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array}$$
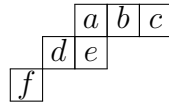. $\qquad\square$

The **type** of a SSYT T is the sequence $\mathrm{type}(T) = (\alpha_1, \alpha_2, \dots)$ where α_i is the number of times that i appears in T. We set

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

Given a pair of partitions $\mu \subseteq \lambda$, the Young diagram of λ/μ is the Young diagram of λ with the Young diagram of μ removed. We define a SSYT of shape λ/μ to be an assignment of positive integers to the boxes of this Young diagram which is weakly increasing in rows and strictly increasing in columns.

Example 4.1.2. If $\lambda = (5, 3, 1)$ and $\mu = (2, 1)$, then

$$\begin{array}{ccc} & & \begin{array}{|c|c|c|} \hline a & b & c \\ \hline \end{array} \\ & \begin{array}{|c|c|} \hline d & e \\ \hline \end{array} \\ \begin{array}{|c|} \hline f \\ \hline \end{array} \end{array}$$

is a SSYT if

- $a \leq b \leq c$,
- $d \leq e$, and
- $a < e$. $\qquad\square$

We define the type of T and x^T in the same way.

Given a partition λ, the **Schur function** s_λ is defined by

$$s_\lambda = \sum_T x^T$$

where the sum is over all SSYT of shape λ. Similarly, given $\mu \subseteq \lambda$, the **skew Schur function** $s_{\lambda/\mu}$ is defined by

$$s_{\lambda/\mu} = \sum_T x^T$$

where the sum is over all SSYT of shape λ/μ. Note that this is a strict generalization of the first definition since we can take $\mu = \varnothing$, the unique partition of 0.

We can make the same definitions in finitely many variables $x_1, \dots, x_n$ if we restrict the sums to be only over SSYT that only use the numbers $1, \dots, n$.

Example 4.1.3. $s_{1,1}(x_1, x_2, \dots, x_n)$ is the sum over SSYT of shape $(1, 1)$. This is the same as a choice of $1 \leq i < j \leq n$, so $s_{1,1}(x_1, \dots, x_n) = \sum_{1 \leq i < j \leq n} x_i x_j = e_2(x_1, \dots, x_n)$, and by the same reasoning, $s_{1,1} = e_2$ in infinitely many variables. More generally, $s_{1^k} = e_k$ for any k.

Also, $s_k = h_k$ since a SSYT of shape (k) is a choice of $i_1 \leq i_2 \leq \cdots \leq i_k$.

For something different, consider $s_{2,1}(x_1, x_2, x_3)$. There are 8 SSYT that of shape $(2, 1)$ that only use $1, 2, 3$:

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$$

From this, we can read off that $s_{2,1}(x_1, x_2, x_3)$ is a symmetric polynomial. Furthermore, it is $m_{2,1}(x_1, x_2, x_3) + 2m_{1,1,1}(x_1, x_2, x_3)$. □

Theorem 4.1.4. *For any $\mu \subseteq \lambda$, the skew Schur function $s_{\lambda/\mu}$ is a symmetric function.*

Proof. We need to check that for every sequence α and every permutation σ, the number of SSYT of shape λ/μ and type α is the same as the number of SSYT of shape λ/μ and type $\sigma(\alpha)$. Since α has only finitely many nonzero entries, we can always replace σ by a permutation σ' that permutes finitely many elements of $\{1, 2, \dots\}$ so that $\sigma(\alpha) = \sigma'(\alpha)$. But then σ' can be written as a finite product of adjacent transpositions $(i, i+1)$. So it's enough to check the case when $\sigma = (i, i+1)$.

Let T be a SSYT of shape λ/μ and type α. Do the following: take the set of columns that only contain exactly one of i or $i+1$. Now consider just the entries of these columns that contain i or $i+1$. The result is a series of isolated rows. In a given row, if there are a instances of i and b instances of $i+1$, then replace it by b instances of i and a instances of $i+1$. The result is still a SSYT, but the type is now $(i, i+1)(\alpha)$. This is reversible, so defines the desired bijection. □

We now focus on Schur functions. Suppose λ is a partition of n. Let $K_{\lambda,\alpha}$ be the number of SSYT of shape λ and type α, this is called a **Kostka number**. The previous theorem says $K_{\lambda,\alpha} = K_{\lambda,\sigma(\alpha)}$ for any permutation σ, so it's enough to study the case when α is a partition. By the definition of Schur function, we have

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu.$$

An important special case is when $\mu = 1^n$. Then $K_{\lambda,1^n}$ is the number of SSYT that use each of the numbers $1, \dots, n$ exactly once. Such a SSYT is called a **standard Young tableau**, and $K_{\lambda,1^n}$ is denoted f^λ.

Theorem 4.1.5. *If $K_{\lambda,\mu} \neq 0$, then $\mu \leq \lambda$ (dominance order). Also, $K_{\lambda,\lambda} = 1$. In particular, the s_λ form a basis for Λ.*

Proof. Suppose that $K_{\lambda,\mu} \neq 0$. Pick a SSYT T of shape λ and type μ. Each number k can only appear in the first k rows of T: otherwise, there is a column with entries $1 \leq i_1 < i_2 < \cdots < i_r < k$ where $r \geq k$, which is a contradiction. This implies that $\mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k$, so $\mu \leq \lambda$.

The only SSYT of shape λ and type λ is the one that fills row i with the number i.

Now the last statement follows from Lemma 2.1.2. □

Corollary 4.1.6. $\{s_\lambda \mid |\lambda| = d\}$ *is a basis for Λ_d.*

Corollary 4.1.7. $\{s_\lambda(x_1, \dots, x_n) \mid |\lambda| = d, \ \ell(\lambda) \leq n\}$ *is a basis for $\Lambda(n)_d$.*

Proof. Note that if $\ell(\lambda) > n$, there are no SSYT only using $1, \dots, n$, so $s_\lambda(x_1, \dots, x_n) = 0$. Hence the set in question spans $\Lambda(n)_d$. Since $\Lambda(n)_d$ is free of rank equal to the size of this set, it must also be a basis. □

4.2. **RSK algorithm.** The RSK algorithm converts a matrix with non-negative integer entries into a pair of SSYT of the same shape. This has a number of remarkable properties which we can use to get identities for Schur functions. The basic step is called row insertion, which we now define.
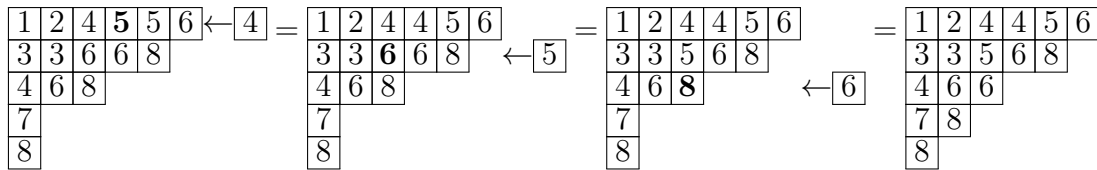
Let T be a SSYT of a partition λ, and let $k \geq 1$ be an integer. The **row insertion**, denoted $T \leftarrow k$, is another tableau defined as follows:

- Find the largest index i such that $T_{1,i-1} \leq k$ (if no such index exists, set $i = 1$).
- Replace $T_{1,i}$ with k and set $k' = T_{1,i}$; we say that k is **bumping** k'. If $i = \lambda_1 + 1$, we are putting k at the end of the row, and the result is $T \leftarrow k$.
- Otherwise, let T' be the SSYT obtained by removing the first row of T. Calculate $T' \leftarrow k'$ and then add the new first row of T back to the result to get $T \leftarrow k$.

Let $I(T \leftarrow k)$ be the set of coordinates of the boxes that get replaced; this is the **insertion path**.

Example 4.2.1. Let $T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 5 & 6 \\ \hline 3 & 3 & 6 & 6 & 8 \\ \cline{1-5} 4 & 6 & 8 \\ \cline{1-3} 7 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array}$ and $k = 4$.

We list the steps below, each time bolding the entry that gets replaced.

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & \mathbf{5} & 5 & 6 \\ \hline 3 & 3 & 6 & 6 & 8 \\ \cline{1-5} 4 & 6 & 8 \\ \cline{1-3} 7 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \leftarrow \boxed{4} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 4 & 5 & 6 \\ \hline 3 & 3 & \mathbf{6} & 6 & 8 \\ \cline{1-5} 4 & 6 & 8 \\ \cline{1-3} 7 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \leftarrow \boxed{5} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 4 & 5 & 6 \\ \hline 3 & 3 & 5 & 6 & 8 \\ \cline{1-5} 4 & 6 & \mathbf{8} \\ \cline{1-3} 7 \\ \cline{1-1} 8 \\ \cline{1-1} \end{array} \leftarrow \boxed{6} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 4 & 5 & 6 \\ \hline 3 & 3 & 5 & 6 & 8 \\ \cline{1-5} 4 & 6 & 6 \\ \cline{1-3} 7 & 8 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$$

The insertion path is $I(T \leftarrow 4) = \{(1,4), (2,3), (3,3), (4,2)\}$. $\qquad\square$

Proposition 4.2.2. $T \leftarrow k$ is a SSYT.

Proof. By construction, the rows of $T \leftarrow k$ are weakly increasing. Also by construction, at each step, if we insert a into row i, then it can only bump a value b with $b > a$. We claim that if $(i,j), (i+1,j') \in I(T \leftarrow k)$, then $j \geq j'$. If not, then $T_{i+1,j} < T_{i,j}$ since otherwise b would bump the number in position $(i+1,j)$ or further left instead of bumping the number in position $(i+1,j')$, but this contradicts that T is a SSYT. In particular, $b = T_{i,j} \leq T_{i,j'}$ and also $T_{i+2,j'} > T_{i+1,j'} > b$, so inserting b into position $(i+1,j')$ preserves the property of being a SSYT. $\qquad\square$

Now we're ready to give the RSK (Robinson–Schensted–Knuth) algorithm. Let A be an infinite matrix (whose rows and columns are indexed by positive integers) with non-negative integer entries (only finitely many of which are nonzero). In examples, we will represent A by a finite northwestern corner which contains all of its positive entries. Create a multiset of tuples (i,j) where the number of times that (i,j) appears is $A_{i,j}$. Now sort them by lexicographic order and put them as the columns of a matrix w_A with 2 rows.

Example 4.2.3. If $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, then $w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 3 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}$. $\qquad\square$

Given A, we're going to create a pair of tableaux (P, Q) by induction as follows. Start with $P(0) = \varnothing$, $Q(0) = \varnothing$. Assuming $P(t)$ and $Q(t)$ are defined, let $P(t+1) = (P(t) \leftarrow (w_A)_{2,t+1})$. Now $P(t+1)$ has a new box that $P(t)$ does not have; add that same box to $Q(t)$ with value $(w_A)_{1,t+1}$ to get $Q(t+1)$. When finished, the result is P (the **insertion tableau**) and Q (the **recording tableau**).

Example 4.2.4. Continuing the previous example, we list $P(t), Q(t)$ in the rows of the following table.

$P(t)$	$Q(t)$
`1`	`1`
`1 1`	`1 1`
`1 1 3`	`1 1 1`
`1 1 2` / `3`	`1 1 1` / `2`
`1 1 2 2` / `3`	`1 1 1 2` / `2`
`1 1 2 2 2` / `3`	`1 1 1 2 2` / `2`
`1 1 2 2 2 3` / `3`	`1 1 1 2 2 2` / `2`
`1 1 2 2 2 3 3` / `3`	`1 1 1 2 2 2 3` / `2`

The last row has the tableaux P and Q. $\qquad\square$

Lemma 4.2.5. *P and Q are both SSYT.*

Proof. P is built by successive row insertions into SSYT, so is itself a SSYT by Proposition 4.2.2. The numbers are put into Q in weakly increasing order since the first row of w_A is weakly increasing. Hence the entries of Q are weakly increasing in each row and also in each column. So it suffices to check no column has repeated entries, and this follows from the next lemma. $\qquad\square$

Lemma 4.2.6. *Let T be a SSYT and $j \leq k$. Then $I(T \leftarrow j)$ is strictly to the left of $I((T \leftarrow j) \leftarrow k)$, i.e., if $(r,s) \in I(T \leftarrow j)$ and $(r,s') \in I((T \leftarrow j) \leftarrow k)$, then $s < s'$. Furthermore, $\#I(T \leftarrow j) \geq \#I((T \leftarrow j) \leftarrow k)$.*

Proof. When inserting k into the first row of $T \leftarrow j$, k must bump a number strictly larger than itself, so gets put in a position strictly to the right of whatever was bumped by j when computing $T \leftarrow j$. The numbers j' and k' that got bumped by j and k satisfy $j' \leq k'$, so we can deduce the first statement by induction on the number of rows.

For the second statement, let $r = \#I(T \leftarrow j)$, so that the last move in computing $T \leftarrow j$ was to add an element to the end of row r. If $\#I((T \leftarrow j) \leftarrow k) \geq r$, then the bump in row r happens strictly to the right of row r, which means an element was added to the end, and hence $r = \#I((T \leftarrow j) \leftarrow k)$ in this case. $\qquad\square$

Theorem 4.2.7. *The RSK algorithm gives a bijection between non-negative integer matrices A with finitely many nonzero entries and pairs of SSYT (P,Q) of the same shape. Furthermore, j appears in P exactly $\sum_i A_{i,j}$ times, while i appears in Q exactly $\sum_j A_{i,j}$ times.*

Proof. The last statement is clear from our construction, so it suffices to prove that RSK gives a bijection. We just give a sketch. First, we can recover w_A from (P,Q) as follows: the last entry in the first row of w_A is the largest entry in Q, and it was added wherever the rightmost occurrence of that entry is. Remove it to get Q'. Now, consider the number in the same position in P. We reverse the row insertion procedure; whatever pops out at the end is the last entry in the second row of w_A. We can undo the row insertion procedure to

get this entry out of P and get a resulting P'. Now repeat to get the rest of the columns of w_A. So the RSK algorithm is injective. We can do this procedure to any pair (P, Q), though we don't know it leads to w_A for some matrix A; surjectivity amounts to proving this is the case. We will skip this check, which amounts to reversing some of the arguments presented above. □

Corollary 4.2.8 (Cauchy identity)**.**

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y)$$

where the sum is over all partitions.

Proof. Given a non-negative integer matrix A with finitely many nonzero entries, assign to it the monomial $m(A) = \prod_{i,j}(x_i y_j)^{A_{i,j}}$. The left hand side is then $\sum_A m(A)$ since the $A_{i,j}$ can be chosen arbitrarily. Via the RSK correspondence, A goes to a pair of SSYT (P, Q), and by Theorem 4.2.7, $m(A) = x^Q y^P$, and so $\sum_A m(A) = \sum_\lambda s_\lambda(x) s_\lambda(y)$. □

Corollary 4.2.9. *The Schur functions form an orthonormal basis with respect to $\langle, \rangle$, i.e., $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.*

Proof. Immediate from the Cauchy identity and Lemma 3.7.1. □

Corollary 4.2.10. *We have*

$$h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda.$$

Proof. Write $h_\mu = \sum_\lambda a_{\lambda,\mu} s_\lambda$ for some coefficients a. By Corollary 4.2.9, $a_{\lambda,\mu} = \langle h_\mu, s_\lambda \rangle$. By definition, we have $s_\lambda = \sum_\nu K_{\lambda,\nu} m_\nu$. But also by definition and Corollary 3.7.3, $\langle h_\mu, m_\nu \rangle = \delta_{\mu,\nu}$. Hence, $a_{\lambda,\mu} = K_{\lambda,\mu}$. □

An important symmetry of the RSK algorithm is the following, but we omit the proof.

Theorem 4.2.11. *If $A \mapsto (P, Q)$ under RSK, then $A^T \mapsto (Q, P)$. In particular, RSK gives a bijection between symmetric non-negative integer matrices with finitely many nonzero entries and the set of all SSYT.*

4.3. **Dual RSK algorithm.** There is a variant of the RSK algorithm for $(0, 1)$-matrices called **dual RSK**. The change occurs in the definition of row insertion: instead of k bumping the leftmost entry that is $> k$, it bumps the leftmost entry that is $\geq k$. This can be analyzed like the RSK algorithm, but we will omit this and state its consequences. Below, a tableau will simply mean an assignment of the positive integers to the boxes of some Young diagram. Furthermore, if P is a tableau, then $P^\dagger$ means the tableau of the transpose of the shape of P with the obvious correspondence of values.

Theorem 4.3.1. *The dual RSK algorithm gives a bijection between $(0, 1)$-matrices A with finitely many nonzero entries and pairs (P, Q) where P and Q are tableaux of the same shape, and $P^\dagger$ and Q are SSYT. Furthermore, the type of P is given by the column sums of A and the type of Q is given by the row sums of A.*

Corollary 4.3.2 (Dual Cauchy identity)**.**

$$\prod_{i,j}(1 + x_i y_j) = \sum_\lambda s_\lambda(x) s_{\lambda^\dagger}(y).$$

Lemma 4.3.3. *Let ω_y be the action of ω on the second copy of Λ in $\Lambda \otimes \Lambda$. Then*

$$\prod_{i,j}(1 + x_iy_j) = \omega_y \prod_{i,j}(1 - x_iy_j)^{-1}.$$

Proof. We have

$$\omega_y \prod_{i,j}(1 - x_iy_j)^{-1} = \omega_y \sum_\lambda m_\lambda(x)h_\lambda(y)$$
$$= \sum_\lambda m_\lambda(x)e_\lambda(y)$$

where the first equality is Proposition 3.7.2 and the second is Theorem 3.4.1. Now we follow the proof of Proposition 3.7.2:

$$\prod_i \prod_j(1 + x_iy_j) = \prod_i \sum_n e_n(y)x_i^n = \sum_\lambda m_\lambda(x)e_\lambda(y).$$

Combining these two gives the result. □

Corollary 4.3.4. $\omega(s_\lambda) = s_{\lambda^\dagger}$.

Proof. Corollary 4.2.8 yields

$$\omega_y \sum_\lambda s_\lambda(x)s_\lambda(y) = \omega_y \prod_{i,j}(1 - x_iy_j)^{-1} = \prod_{i,j}(1 + x_iy_j) = \sum_\lambda s_\lambda(x)s_{\lambda^\dagger}(y).$$

The $s_\lambda(x)$ are linearly independent, so $\omega_y(s_\lambda(y)) = s_{\lambda^\dagger}(y)$. □

4.4. **Determinantal formula.** For this section, we will fix a positive integer n. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a non-negative integer sequence. Define

$$a_\alpha = \det(x_i^{\alpha_j})_{i,j=1}^n = \det \begin{pmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots & x_1^{\alpha_n} \\ x_2^{\alpha_1} & x_2^{\alpha_2} & \cdots & x_2^{\alpha_n} \\ \vdots & & & \vdots \\ x_n^{\alpha_1} & x_n^{\alpha_2} & \cdots & x_n^{\alpha_n} \end{pmatrix}.$$

Note that a_α is *skew-symmetric*: if we permute a_α by a permutation $\sigma \in \mathfrak{S}_n$, then it changes by $\mathrm{sgn}(\sigma)$. Let $\rho = (n-1, n-2, \ldots, 1, 0)$.

Lemma 4.4.1. (a) $\prod_{1 \le i < j \le n}(x_i - x_j)$ *divides every skew-symmetric polynomial in* $x_1, \ldots, x_n$. (b) $a_\rho = \prod_{1 \le i < j \le n}(x_i - x_j)$.

Proof. (a) Let $f(x_1, \ldots, x_n)$ be skew-symmetric and let σ be the transposition (i, j). Then $\sigma f = -f$. However, σf and f are the same if we replace x_j by x_i, so this says that specializing x_j to x_i gives 0, i.e., f is divisible by $(x_i - x_j)$. This is true for any i, j, so this proves (a).

(b) a_ρ is divisible by $\prod_{1 \le i < j \le n}(x_i - x_j)$ since it is skew-symmetric. But also note that both are polynomials of degree $1 + 2 + \cdots + (n-1) = \binom{n}{2}$, so they are equal up to some integer multiple. The coefficient of $x_1^{n-1}x_2^{n-2} \cdots x_{n-1}$ for both is 1, so they are actually the same. □

Define $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

Given $\nu \subseteq \lambda$, let $K_{\lambda/\nu,\mu}$ be the number of SSYT of skew shape λ/ν and type μ.

Lemma 4.4.2. *Let μ, ν be partitions with $\ell(\mu) \leq n$ and $\ell(\nu) \leq n$. Then*

$$a_{\nu+\rho} e_\mu(x_1, \ldots, x_n) = \sum_\lambda K_{\lambda^\dagger/\nu^\dagger, \mu} a_{\lambda+\rho}.$$

Proof. Throughout this proof, all symmetric functions are understood to be specialized to the variables $x_1, \ldots, x_n$. First, we claim that given a partition μ, the coefficient of $x^{\lambda+\rho}$ in $a_{\nu+\rho} e_\mu$ is $K_{\lambda^\dagger/\nu^\dagger, \mu}$. To get a monomial in $a_\rho e_{\mu_1} \cdots e_{\mu_k}$, we pick a monomial x^β in $a_{\nu+\rho}$ and successively multiply it by monomials $x^{\alpha(1)}, \ldots, x^{\alpha(k)}$ where $x^{\alpha(i)}$ is taken from e_{μ_i}. Note that each partial product $a_{\nu+\rho} e_{\mu_1} \cdots e_{\mu_r}$ is skew-symmetric, so each of its monomials have distinct exponents on all of the variables. So, we're only interested in choosing $x^{\alpha(r+1)}$ so that the product $x^\beta x^{\alpha(1)} \cdots x^{\alpha(r+1)}$ has all exponents distinct. Since $x^{\alpha(r+1)}$ is a product of distinct variables, the relative order of the exponents remains the same. Since we're interested in the coefficient of $x^{\lambda+\rho}$, whose exponents are strictly decreasing, we can only get to this if β is strictly decreasing and the $x^{\alpha(r+1)}$ is chosen so that the exponents of $x^\beta x^{\alpha(1)} \cdots x^{\alpha(r+1)}$ are strictly decreasing. The only β that works is $\nu + \rho$, and so the condition is the same as requiring that $\gamma(r+1) := \nu + \alpha(1) + \cdots + \alpha(r+1)$ is a partition for each r.

Note then we get a sequence of partitions

$$\nu = \gamma(0) \subseteq \gamma(1) \subseteq \gamma(2) \subseteq \cdots \subseteq \gamma(n) = \lambda$$

such that the difference $\gamma(r+1)/\gamma(r)$ only has boxes in different rows, and that conversely, given such a sequence, we can find a sequence of monomials that corresponds to this. However, this sequence is also equivalently encoding a labeling of the Young diagram of λ/ν which is weakly increasing in columns and strictly increasing in rows, i.e., taking the transpose gives a SSYT of $\lambda^\dagger/\nu^\dagger$ and type μ. So the claim is proven.

Finally, consider the difference

$$a_{\nu+\rho} e_\mu - \sum_\lambda K_{\lambda^\dagger/\nu^\dagger, \mu} a_{\lambda+\rho}.$$

If $\lambda' \neq \lambda$, then the coefficient of $x^{\lambda+\rho}$ in $a_{\lambda'+\rho}$ is 0, so the coefficient of each $x^{\lambda+\rho}$ of this difference is 0. However, any nonzero skew-symmetric function of degree $|\lambda| + \binom{n}{2}$ has a monomial of the form $x^{\lambda+\rho}$ for some partition λ, so we conclude that the difference is 0. $\square$

Corollary 4.4.3. *Given a partition λ,*

$$s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho}.$$

Proof. Again, in this proof, all symmetric functions are understood to be specialized to $x_1, \ldots, x_n$. Take $\nu = \varnothing$ in Lemma 4.4.2 and divide both sides by a_ρ to get

$$e_\mu = \sum_\lambda K_{\lambda^\dagger, \mu} \frac{a_{\lambda+\rho}}{a_\rho}.$$

However, we also have an expression

$$e_\mu = \sum_\lambda K_{\lambda^\dagger, \mu} s_\lambda$$

by applying ω to Corollary 4.2.10. The sets $\{s_\lambda \mid \ell(\lambda) \leq n\}$ and $\{e_\mu \mid \mu_1 \leq n\}$ are both bases of $\Lambda(n)$ (by Corollary 4.1.7 and Theorem 3.3.4), so we can invert the matrix $(K_{\lambda^\dagger, \mu})$ to conclude that $s_\lambda = a_{\lambda+\rho}/a_\rho$. $\square$

Remark 4.4.4. (For those familiar with Lie theory.) The formula above is really an instance of the Weyl character formula for the Lie algebra $\mathfrak{gl}_n(\mathbf{C})$ (or actually, $\mathfrak{sl}_n(\mathbf{C})$ since it's semisimple, but we'll phrase everything in terms of $\mathfrak{gl}_n(\mathbf{C})$ because it's cleaner). To translate, first note that we can evaluate determinants by using a sum over all permutations, and in our case this gives

$$a_\alpha = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma(x^\alpha).$$

In the context of $\mathfrak{gl}_n(\mathbf{C})$, (integral) weights are identified with elements of $\mathbf{Z}^n$, while dominant weights are the weakly decreasing ones. Also, ρ is used here to have the same meaning as in Lie theory: it is the sum of the fundamental dominant weights. Finally, $\mathfrak{S}_n$ is the Weyl group of $\mathfrak{gl}_n(\mathbf{C})$, and $s_\lambda(x_1, \ldots, x_n)$ is the character of the irreducible representation with highest weight λ. Then our formula becomes

$$s_\lambda(x_1, \ldots, x_n) = \frac{\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma(x^{\lambda+\rho})}{\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma(x^\rho)}$$

which is the Weyl character formula, as might be found in [Hu, §24.3]. $\square$

4.5. **Multiplying Schur functions, Pieri rule.**

Corollary 4.5.1. $s_\nu e_\mu = \sum_\lambda K_{\lambda^\dagger/\nu^\dagger,\mu} s_\lambda$.

Proof. By Lemma 4.4.2, we have

$$a_{\nu+\rho}e_\mu(x_1, \ldots, x_n) = \sum_\lambda K_{\lambda^\dagger/\nu^\dagger,\mu} a_{\lambda+\rho}$$

for $n \geq \max(\ell(\lambda), \ell(\nu))$. Divide both sides by a_ρ and use Corollary 4.4.3 to get the desired identity in finitely many variables $x_1, \ldots, x_n$. Since it holds for all $n \gg 0$, it also holds when $n = \infty$. $\square$

Corollary 4.5.2. $s_\nu h_\mu = \sum_\lambda K_{\lambda/\nu,\mu} s_\lambda$.

Proof. Apply ω to Corollary 4.5.1 and use Corollary 4.3.4 to get the desired identity with $\nu^\dagger$ and $\lambda^\dagger$ in place of ν and λ. But that's just an issue of indexing, so we get the desired identity. $\square$

Theorem 4.5.3. *For any $f \in \Lambda$, we have*

$$\langle fs_\nu, s_\lambda \rangle = \langle f, s_{\lambda/\nu} \rangle.$$

Proof. Both sides of the equation are linear in f, so it suffices to prove this when f ranges over a particular basis, and we choose h_μ. By Corollary 4.5.2, $\langle h_\mu s_\nu, s_\lambda \rangle = K_{\lambda/\nu,\mu}$. This is the coefficient of m_μ in $s_{\lambda/\nu}$. Since $\langle h_\nu, m_\mu \rangle = \delta_{\nu,\mu}$, we conclude that $K_{\lambda/\nu,\mu} = \langle h_\mu, s_{\lambda/\nu} \rangle$. $\square$

Of particular note is when $f = s_\mu$. Since the s_λ are a basis, we have unique expressions

$$(4.5.4) \qquad\qquad\qquad s_\mu s_\nu = \sum_\lambda c_{\mu,\nu}^\lambda s_\lambda,$$

and the $c_{\mu,\nu}^\lambda$ are called **Littlewood–Richardson coefficients**. We will see some special cases soon and study this in more depth later. Since the s_λ are an orthonormal basis, we get

$$(4.5.5) \qquad\qquad\qquad c_{\mu,\nu}^\lambda = \langle s_\mu s_\nu, s_\lambda \rangle = \langle s_\mu, s_{\lambda/\nu} \rangle.$$

In particular, we also have an identity

$$s_{\lambda/\nu} = \sum_\mu c^\lambda_{\mu,\nu} s_\mu.$$

From the definition, we have

$$c^\lambda_{\mu,\nu} = c^\lambda_{\nu,\mu}.$$

Applying ω to (4.5.4), we get

(4.5.6) $$c^\lambda_{\mu,\nu} = c^{\lambda^\dagger}_{\nu^\dagger,\mu^\dagger}.$$

We can give an interpretation for the Littlewood–Richardson coefficients in the special case where μ (or ν) has a single part or all parts equal to 1. Say that λ/ν is a **horizontal strip** if no column in the skew Young diagram of λ/ν contains 2 or more boxes. Similarly, say that λ/ν is a **vertical strip** if no row in the skew Young diagram of λ/ν contains 2 or more boxes.

Theorem 4.5.7 (Pieri rule). • *If $\mu = (1^k)$, then*

$$c^\lambda_{(1^k),\nu} = \begin{cases} 1 & \text{if } |\lambda| = |\nu| + k \text{ and } \lambda/\nu \text{ is a vertical strip} \\ 0 & \text{otherwise} \end{cases}.$$

In other words,

$$s_\nu s_{1^k} = \sum_\lambda s_\lambda$$

where the sum is over all λ such that λ/ν is a vertical strip of size k.
• *If $\mu = (k)$, then*

$$c^\lambda_{(k),\nu} = \begin{cases} 1 & \text{if } |\lambda| = |\nu| + k \text{ and } \lambda/\nu \text{ is a horizontal strip} \\ 0 & \text{otherwise} \end{cases}.$$

In other words,

$$s_\nu s_k = \sum_\lambda s_\lambda$$

where the sum is over all λ such that λ/ν is a horizontal strip of size k.

Proof. Since $s_{1^k} = e_k$, we have $s_\nu s_{1^k} = \sum_\lambda K_{\lambda^\dagger/\nu^\dagger,k} s_\lambda$ by Corollary 4.5.1. So $c^\lambda_{1^k,\nu}$ is the number of SSYT of shape $\lambda^\dagger/\nu^\dagger$ using k 1's. There is at most one such SSYT, and it exists exactly when $|\lambda/\nu| = k$ and no two boxes of $\lambda^\dagger/\nu^\dagger$ are in the same column, i.e., λ/ν is a vertical strip.

The proof the second identity is similar, or can be obtained by using ω. □

Example 4.5.8. To multiply s_λ by s_k, it suffices to enumerate all partitions that we can get by adding k boxes to the Young diagram of λ, no two of which are in the same column. For example, here we have drawn all such ways to add 2 boxes to $(4,2)$:
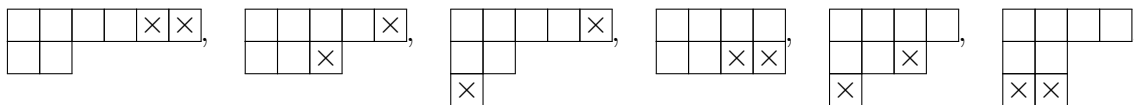
So

$$s_{4,2} s_2 = s_{6,2} + s_{5,3} + s_{5,2,1} + s_{4,4} + s_{4,3,1} + s_{4,2,2}.$$ □

Recall that for $|\lambda| = n$, $f^\lambda = K_{\lambda,1^n}$ is the number of standard Young tableaux of shape λ.

Corollary 4.5.9. $s_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda$.

Proof. The Pieri rule says that to multiply s_1^n, we first enumerate all sequences $\lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(n)}$ where $|\lambda^{(i)}| = i$. Then the result is the sum of s_λ with multiplicity given by the number of sequences with $\lambda^{(n)} = \lambda$. But such sequences are in bijection with standard Young tableaux: label the unique box in $\lambda^{(i)}/\lambda^{(i-1)}$ with i. $\qquad \square$

Theorem 4.5.10. $\omega(s_{\lambda/\nu}) = s_{\lambda^\dagger/\nu^\dagger}$.

Proof. First, we have

$$
\begin{aligned}
\langle s_{\mu^\dagger}, s_{\lambda^\dagger/\nu^\dagger} \rangle &= \langle s_{\mu^\dagger} s_{\nu^\dagger}, s_{\lambda^\dagger} \rangle && \text{(Theorem 4.5.3)} \\
&= \langle \omega(s_\mu s_\nu), \omega(s_\lambda) \rangle && \text{(Corollary 4.3.4)} \\
&= \langle s_\mu s_\nu, s_\lambda \rangle && \text{(Corollary 3.7.5)} \\
&= \langle s_\mu, s_{\lambda/\nu} \rangle && \text{(Theorem 4.5.3)} \\
&= \langle \omega(s_\mu), \omega(s_{\lambda/\nu}) \rangle && \text{(Corollary 3.7.5)} \\
&= \langle s_{\mu^\dagger}, \omega(s_{\lambda/\nu}) \rangle. && \text{(Corollary 4.3.4)}
\end{aligned}
$$

The pairing is nondegenerate, so if we fix λ, ν and allow μ to vary, we get $s_{\lambda^\dagger/\nu^\dagger} = \omega(s_{\lambda/\nu})$. $\qquad \square$

4.6. **Jacobi–Trudi identity.** Corollary 4.5.2 and Corollary 4.5.1 with $\nu = \varnothing$ explain how to rewrite the h_μ and e_μ bases in terms of the Schur basis using Kostka numbers. The Jacobi–Trudi identities go the other way around. We'll do something more general with skew Schur functions though.

Theorem 4.6.1. *Pick $\mu \subseteq \lambda$ with $\ell(\lambda) \leq n$. Set $h_i = 0$ if $i < 0$. Then*

$$
s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j})_{i,j=1}^n = \det \begin{pmatrix}
h_{\lambda_1 - \mu_1} & h_{\lambda_1 - \mu_2 + 1} & h_{\lambda_1 - \mu_3 + 2} & \cdots & h_{\lambda_1 - \mu_n + n - 1} \\
h_{\lambda_2 - \mu_1 - 1} & h_{\lambda_2 - \mu_2} & h_{\lambda_2 - \mu_3 + 1} & \cdots & h_{\lambda_2 - \mu_n + n - 2} \\
\vdots & & & & \vdots \\
h_{\lambda_n - \mu_1 - n + 1} & h_{\lambda_n - \mu_2 - n + 2} & h_{\lambda_n - \mu_3 - n + 3} & \cdots & h_{\lambda_n - \mu_n}
\end{pmatrix}
$$

$$
s_{\lambda/\mu} = \det(e_{\lambda_i^\dagger - \mu_j^\dagger - i + j})_{i,j=1}^n = \det \begin{pmatrix}
e_{\lambda_1^\dagger - \mu_1^\dagger} & e_{\lambda_1^\dagger - \mu_2^\dagger + 1} & e_{\lambda_1^\dagger - \mu_3^\dagger + 2} & \cdots & e_{\lambda_1^\dagger - \mu_n^\dagger + n - 1} \\
e_{\lambda_2^\dagger - \mu_1^\dagger - 1} & e_{\lambda_2^\dagger - \mu_2^\dagger} & e_{\lambda_2^\dagger - \mu_3^\dagger + 1} & \cdots & e_{\lambda_2^\dagger - \mu_n^\dagger + n - 2} \\
\vdots & & & & \vdots \\
e_{\lambda_n^\dagger - \mu_1^\dagger - n + 1} & e_{\lambda_n^\dagger - \mu_2^\dagger - n + 2} & e_{\lambda_n^\dagger - \mu_3^\dagger - n + 3} & \cdots & e_{\lambda_n^\dagger - \mu_n^\dagger}.
\end{pmatrix}
$$

Proof. First, note that if we move from n to $n + 1$, the determinant does not change because the new row added is $(0, 0, \ldots, 0, 1)$. Fix μ and work in $\mathbf{Z}[\![x_1, \ldots, x_N, y_1, \ldots, y_N]\!]$ where $N \geq n$. We have

$$
\begin{aligned}
\sum_\lambda s_{\lambda/\mu}(x) s_\lambda(y) &= \sum_\lambda \sum_\nu c_{\mu,\nu}^\lambda s_\nu(x) s_\lambda(y) && \text{(by (4.5.5))} \\
&= \sum_\nu s_\nu(x) s_\nu(y) s_\mu(y) && \text{(by (4.5.5))} \\
&= s_\mu(y) \prod_{i,j} (1 - x_i y_j)^{-1} && \text{(Corollary 4.2.8)} \\
&= s_\mu(y) \sum_\nu h_\nu(x) m_\nu(y). && \text{(Proposition 3.7.2)}
\end{aligned}
$$

Multiply both sides by $a_\rho(y)$ to get

$$\sum_\lambda s_{\lambda/\mu}(x)a_{\lambda+\rho}(y) = a_{\mu+\rho}(y)\sum_\nu h_\nu(x)m_\mu(y) \hspace{2cm} \text{(Corollary 4.4.3)}$$

$$= \left(\sum_{\sigma\in\mathfrak{S}_N} \text{sgn}(\sigma)\sigma(y^{\mu+\rho})\right)\left(\sum_{\alpha\in\mathbf{Z}_{\geq 0}^N} h_\alpha(x)y^\alpha\right)$$

$$= \sum_{\sigma\in\mathfrak{S}_N}\sum_\alpha \text{sgn}(\sigma)h_\alpha(x)y^{\alpha+\sigma(\mu+\rho)}.$$

Now take the coefficient of $y^{\lambda+\rho}$. The left hand side gives $s_{\lambda/\mu}(x)$, while the right hand side gives

$$\sum_{\sigma\in\mathfrak{S}_N} \text{sgn}(\sigma)h_{\lambda+\rho-\sigma(\mu+\rho)}(x) = \det(h_{\lambda_i-\mu_j-i+j})_{i,j=1}^N.$$

So we get the desired identity in N variables; let $N\to\infty$ to get it in general.

The second identity follows from the first by applying ω. $\qquad\square$

Remark 4.6.2. It is possible to give an elegant combinatorial proof of the Jacobi–Trudi identity by interpreting SSYT as non-crossing lattice paths and using the Gessel–Viennot method of enumerating non-crossing lattice paths. The interested reader can find this argument in [Sta, §7.16]. $\qquad\square$

5. Representation theory of the symmetric groups

5.1. **Symmetric groups.** We now focus on the symmetric groups $G = \mathfrak{S}_n$. Recall that in §1.6.4, we showed that all of the characters of its representations are integer-valued. We define CF_n to be the space of rational-valued class functions on $\mathfrak{S}_n$. Recall that the pairing on CF_n is given by

$$(\varphi,\psi)_{\mathfrak{S}_n} = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\varphi(\sigma)\psi(\sigma)$$

(there is no complex conjugation since they are rational valued).

First, every permutation $\sigma\in\mathfrak{S}_n$ has a decomposition as a product of disjoint cycles (a cycle, denoted $(i_1,i_2,\ldots,i_k)$, is the permutation which sends i_j to i_{j+1} for $j < k$ and i_k to i_1), and the lengths of these cycles, including cycles of length 1, arranged in decreasing order gives a partition of n, which we denote $t(\sigma)$ and call the **cycle type**.

Recall that for a partition λ, we let $m_i(\lambda)$ denote the number of times that i appears as an entry, and $z_\lambda = \prod_i m_i(\lambda)!i^{m_i(\lambda)}$.

Lemma 5.1.1. *The conjugacy class assigned to λ has size $n!/z_\lambda$.*

Proof. Given a permutation σ of cycle type λ, we claim that the centralizer of σ has size z_λ. To see this, note that a cycle $(i_1,i_2,\ldots,i_k)$ is equal to a cyclic shift $(i_r,i_{r+1},\ldots,i_k,i_1,\ldots,i_{r-1})$. So any τ that sends a cycle of σ to any cyclic shift of another cycle of the same length is in the centralizer, and there are z_λ such τ. $\qquad\square$

Given a partition λ, let 1_λ denote the class function which is 1 on all permutations with cycle type λ and 0 on all other permutations.

Corollary 5.1.2. *Given partitions λ,μ of n, we have $(1_\lambda,1_\mu)_{\mathfrak{S}_n} = z_\lambda^{-1}\delta_{\lambda,\mu}$.*

Proof. If $\lambda \neq \mu$, then the definition of the pairing shows that $(1_\lambda, 1_\mu) = 0$. Otherwise, $(1_\lambda, 1_\lambda) = \frac{1}{n!}c$ where c is the size of the conjugacy class of cycle type λ, which we just said is $n!/z_\lambda$. $\qquad\square$

Next, given non-negative integers n, m, we can think of $\mathfrak{S}_n \times \mathfrak{S}_m$ as a subgroup of $\mathfrak{S}_{n+m}$ if we identify $\mathfrak{S}_n$ with the subgroup which is the identity on $n+1, \ldots, n+m$ and if we identify $\mathfrak{S}_m$ with the subgroup which is the identity on $1, \ldots, n$. Define an induction product

$$\circ \colon \mathrm{CF}_n \times \mathrm{CF}_m \to \mathrm{CF}_{n+m}$$

$$\varphi \circ \psi = \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}}(\varphi \otimes \psi).$$

This turns $\mathrm{CF} := \bigoplus_{n \geq 0} \mathrm{CF}_n$ into a commutative ring (though it requires verification). Our next task is to show that it is isomorphic to $\Lambda_{\mathbf{Q}}$ (which automatically implies that the product is commutative).

5.2. **The characteristic map.** The **Frobenius characteristic map** is the linear function

$$\mathrm{ch} \colon \mathrm{CF}_n \to \Lambda_{\mathbf{Q},n}$$

$$\mathrm{ch}(\varphi) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varphi(\sigma) p_{t(\sigma)}$$

where recall that p is the power sum symmetric function. Alternatively, if we set $\varphi(\lambda)$ to be the value of φ on any permutation with cycle type λ, then $\mathrm{ch}(\varphi) = \sum_\lambda z_\lambda^{-1} \varphi(\lambda) p_\lambda$ by Lemma 5.1.1. Put these together to define a linear function

$$\mathrm{ch} \colon \bigoplus_{n \geq 0} \mathrm{CF}_n \to \Lambda_{\mathbf{Q}}.$$

Proposition 5.2.1. ch *is an isometry, i.e., given* $\varphi, \psi \in \mathrm{CF}_n$,

$$(\varphi, \psi)_{\mathfrak{S}_n} = \langle \mathrm{ch}(\varphi), \mathrm{ch}(\psi) \rangle.$$

Proof. Given that the conjugacy class of λ has size $n!/z_\lambda$, we have

$$\langle \mathrm{ch}(\varphi), \mathrm{ch}(\psi) \rangle = \langle \sum_\lambda z_\lambda^{-1} \varphi(\lambda) p_\lambda, \sum_\mu z_\mu^{-1} \psi(\mu) p_\mu \rangle$$

$$= \sum_\lambda z_\lambda^{-1} \varphi(\lambda) \psi(\lambda)$$

$$= (\varphi, \psi)_{\mathfrak{S}_n},$$

where the second equality is orthogonality of the p_λ (Proposition 3.7.4) and the third equality uses that φ, ψ are constant on conjugacy classes. $\qquad\square$

Proposition 5.2.2. ch *is a ring isomorphism, i.e., given* $\varphi \in \mathrm{CF}_n$ *and* $\psi \in \mathrm{CF}_m$, *we have*

$$\mathrm{ch}(\varphi \circ \psi) = \mathrm{ch}(\varphi)\mathrm{ch}(\psi).$$

Proof. For two partitions λ, μ, write $\lambda \cup \mu$ for the partition with the combined parts of λ and μ sorted in order.

We claim that $1_\lambda \circ 1_\mu = \frac{z_{\lambda \cup \mu}}{z_\lambda z_\mu} 1_{\lambda \cup \mu}$. To see this, let ν be any partition of $n+m$. Then by Frobenius reciprocity and Corollary 5.1.2,

$$(\mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}}(1_\lambda \otimes 1_\mu), 1_\nu)_{\mathfrak{S}_{n+m}} = (1_\lambda \otimes 1_\mu, \mathrm{Res}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} 1_\nu)_{\mathfrak{S}_n \times \mathfrak{S}_m} = \frac{\delta_{\lambda \cup \mu, \nu}}{z_\lambda z_\mu}.$$

Hence $1_\lambda \circ 1_\mu = c1_{\lambda \cup \mu}$ where $c = \frac{(1_\lambda \circ 1_\mu, 1_{\lambda \cup \mu})}{(1_{\lambda \cup \mu}, 1_{\lambda \cup \mu})} = \frac{z_{\lambda \cup \mu}}{z_\lambda z_\mu}$.

Next, $\mathrm{ch}(1_\lambda) = p_\lambda / z_\lambda$, so we see that $\mathrm{ch}(1_\lambda \circ 1_\mu) = \mathrm{ch}(1_\lambda)\mathrm{ch}(1_\mu)$. Since the 1_λ form a basis for CF, we conclude that ch is a ring homomorphism. Finally, the 1_λ map to a basis for $\Lambda_{\mathbf{Q}}$, so we also conclude that it is an isomorphism. $\qquad \square$

We now wish to get a more refined statement. Let $R_n \subset \mathrm{CF}_n$ be the $\mathbf{Z}$-submodule of virtual characters, i.e., integer linear combinations of characters, and set $R = \bigoplus_{n \geq 0} R_n$. Our goal is to determine the irreducible characters of $\mathfrak{S}_n$.

Proposition 5.2.3. *Suppose $\varphi_1, \varphi_2, \ldots, \varphi_{p(n)} \in R_n$ form an orthonormal basis with respect to $(,)_{\mathfrak{S}_n}$. Then the irreducible characters are $\varepsilon_1 \varphi_1, \varepsilon_2 \varphi_2, \ldots, \varepsilon_{p(n)} \varphi_{p(n)}$ for some choices $\varepsilon_i \in \{1, -1\}$.*

Proof. By definition, the irreducible characters belong to R_n and they form an orthonormal basis by Theorem 1.5.2. Now write the φ_i as integer linear combinations of the irreducible characters. These coefficients give an orthogonal matrix (with respect to a positive definite form) with integer entries. The only orthonormal vectors with integer entries are standard basis vectors and their negatives, so each row is one of these. Since the matrix is invertible, we see that the matrix has exactly one nonzero entry in each row and column, and that entry is ± 1. $\qquad \square$

With this in mind, the first step is to find an orthonormal basis of R_n. We know that the Schur functions form an orthonormal basis of Λ, so we define
$$\chi^\lambda = \mathrm{ch}^{-1}(s_\lambda).$$
Our goal now is show that $\chi^\lambda \in R$ and that they in fact are the irreducible characters.

Let $1_{\mathfrak{S}_n}$ denote the trivial homomorphism $\mathfrak{S}_n \to \mathbf{GL}_1(\mathbf{C})$ which sends everything to 1. Its character just assigns 1 to every permutation, so $\mathrm{char}(1_{\mathfrak{S}_n}) = \sum_\lambda 1_\lambda$. For every partition $\alpha = (\alpha_1, \ldots, \alpha_k)$, define
$$\eta^\alpha = 1_{\mathfrak{S}_{\alpha_1}} \circ \cdots \circ 1_{\mathfrak{S}_{\alpha_k}}.$$

Lemma 5.2.4. $\mathrm{ch}(\eta^\alpha) = h_\alpha$.

Proof. First, $\mathrm{ch}(1_{\mathfrak{S}_n}) = \sum_\lambda z_\lambda^{-1} p_\lambda$, which, by Theorem 3.6.3, is h_n. Now use that ch is a ring homomorphism. $\qquad \square$

Corollary 5.2.5. $\chi^\lambda \in R$ *and* ch *restricts to an isomorphism* $R \to \Lambda$.

Proof. We have $\eta^\alpha \in R$ and $s_\lambda = \det(h_{\lambda_i - i + j})_{i,j=1}^n$ by the Jacobi–Trudi identity (Theorem 4.6.1). This expresses the Schur function as an integer linear combination of h_α, so χ^λ is also an integer linear combination of the η^α and hence $\chi^\lambda \in R$.

Since the s_λ form an orthonormal basis of Λ and ch is an isometry, we conclude that up to a sign, the χ^λ are the irreducible characters by Proposition 5.2.3. In particular, every element of R is an integer linear combination of the χ^λ. Since ch sends them to a basis of Λ, we get the second statement. $\qquad \square$

Finally, we have to determine which of χ^λ and $-\chi^\lambda$ is actually the irreducible character. To do this, we evaluate on the identity element of $\mathfrak{S}_n$. The trace of the identity element is the dimension of the representation, so we just need to determine if this evaluation is positive or negative. It will turn out that $\chi^\lambda(1) > 0$, and more generally, the Murnaghan–Nakayama rule, to be studied next, will determine the evaluation at any permutation.

5.3. **Murnaghan–Nakayama rule.** Given partitions λ, μ, let $\chi^\lambda(\mu)$ denote the evaluation of χ^λ on any permutation with cycle type μ. Then by definition,

$$\chi^\lambda = \sum_\mu \chi^\lambda(\mu) 1_\mu.$$

Applying the characteristic map, we get

$$s_\lambda = \sum_\mu z_\mu^{-1} \chi^\lambda(\mu) p_\mu.$$

So to determine these evaluations, we need to determine how the Schur functions can be written in terms of the power sum symmetric functions. More generally, given $\nu \subseteq \lambda$, define
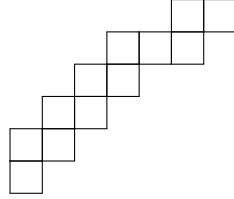
$$\chi^{\lambda/\nu} = \mathrm{ch}^{-1}(s_{\lambda/\nu}),$$

so that

(5.3.1) $$s_{\lambda/\nu} = \sum_\mu z_\mu^{-1} \chi^{\lambda/\nu}(\mu) p_\mu.$$

We will first study the inverse problem of expressing the p's in terms of the Schur functions and get what we want using the scalar product.

Define a **border strip** to be a connected skew diagram with no 2×2 subdiagram. (Sharing only a corner is not considered connected, so $21/1$ is not connected.) Here is an example border strip:

The **height** of a border strip B is denoted $\mathrm{ht}(B)$, and is the number of rows minus 1. In the example above, the height is 5.

Theorem 5.3.2. *Given a positive integer r, we have*

$$s_\mu p_r = \sum_\lambda (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda$$

where the sum is over all λ such that $\mu \subseteq \lambda$ and λ/μ is a border strip of size r.

Proof. It suffices to prove this in n variables where $n \gg 0$. Recall the definition of the determinant $a_\alpha = \det(x_i^{\alpha_j})_{i,j=1}^n$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is any sequence of non-negative integers. Recall $\rho = (n-1, n-2, \ldots, 1, 0)$. Let ε_j be the sequence with a single 1 in position j and 0's elsewhere. By Corollary 4.4.3, we have $s_\lambda = a_{\lambda+\rho}/a_\rho$.

We have

$$a_{\mu+\rho} p_r = \Big(\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) x^{\sigma(\mu+\rho)} \Big) \Big(\sum_{j=1}^n x_j^r \Big) = \sum_{j=1}^n \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) x^{\sigma(\mu+\rho+r\varepsilon_j)} = \sum_{j=1}^n a_{\mu+\rho+r\varepsilon_j},$$

where in the second equality, we multiply $x^{\sigma(\mu+\rho)}$ by $x_{\sigma(j)}^r$ to get $x^{\sigma(\mu+\rho+r\varepsilon_j)}$.

Next, $a_\alpha = -a_\beta$ if $\beta = (\alpha_1, \ldots, \alpha_{i-2}, \alpha_i, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$, i.e., β is obtained from α by swapping two consecutive entries. In particular, $a_{\alpha+\rho} = -a_{\gamma+\rho}$ where $\gamma = (\alpha_1, \ldots, \alpha_{i-2}, \alpha_i -$

$1, \alpha_{i-1} + 1, \alpha_{i+1}, \ldots, \alpha_n)$ and also $a_{\alpha+\rho} = 0$ if $\alpha + \rho$ has any repeating entries. We say that γ is obtained by a shifted transposition at position $i - 1$.

Suppose that $\mu + r\varepsilon_j + \rho$ has no repeating entries. Note that $\mu + r\varepsilon_j/\mu$ is a border strip of size r and height 0 ($\mu + r\varepsilon_j$ may not be a partition, but we will draw its diagram in the expected way). If $\mu + r\varepsilon_j$ is not a partition, then replace it by the shifted transposition at position $j - 1$. What happens in rows $j - 1$ and j is that $(\mu_{j-1}, \mu_j + r)$ gets replaced by $(\mu_j + r - 1, \mu_{j-1} + 1)$. This is a new shape that contains μ and the complement is a border strip with two rows, the first of length $\mu_j + r - 1 - \mu_{j-1}$ and the second of length $\mu_{j-1} + 1 - \mu_j$. If this is a partition, we stop, otherwise we apply another shifted transposition at position $j - 2$, and so on.

The end result is a new partition containing μ whose complement is a border strip. The height of this border strip is precisely the number of shifted transpositions we applied, and all border strips arise in this way by taking j to be the row index of the last row of the border strip (we will not go into detail on this). Hence we get the formula

$$\sum_{j=1}^{n} a_{\mu+\rho+r\varepsilon_j} = \sum_{\lambda}(-1)^{\operatorname{ht}(\lambda/\mu)} a_{\lambda+\rho}$$

where the sum is over all λ containing μ such that λ/μ is a border strip of size r and $\ell(\lambda) \leq n$. If $n \geq \ell(\mu) + r$, this accounts for all possible border strips. Now divide both sides by a_ρ to get the desired identity. $\qquad\square$

Example 5.3.3. $s_1 p_4 = s_5 - s_{3,2} + s_{2,2,1} - s_{1^5}$ corresponding to the following border strips:
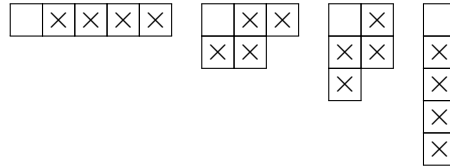
In the notation of the proof, these come from $a_{1+\rho+4\varepsilon_j}$ for $j = 1, 2, 3, 5$. $\qquad\square$

Given a sequence of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_k)$, a **border-strip tableau** of shape λ/μ and of type α is a sequence of partitions $\mu = \lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^k = \lambda$ such that λ^i/λ^{i-1} is a border strip of size α_i. The height of this tableau is the sum of the heights of the border strips λ^i/λ^{i-1} (empty border strips have height 0, rather than -1).

Corollary 5.3.4.

$$s_\mu p_\alpha = \sum_{\lambda}\sum_{T}(-1)^{\operatorname{ht}(T)} s_\lambda$$

where the inner sum is over all border-strip tableaux T of shape λ/μ and type α. In particular,

$$p_\alpha = \sum_{\lambda}\sum_{T}(-1)^{\operatorname{ht}(T)} s_\lambda$$

where the inner sum is over all border-strip tableaux T of shape λ and type α.

Corollary 5.3.5. $\chi^{\lambda/\nu}(\mu) = \sum_T(-1)^{\operatorname{ht}(T)}$ where the sum is over all border-strip tableaux T of shape λ/ν and type μ.

Proof. By (5.3.1) and orthogonality properties of the p_μ, we get

$$
\begin{aligned}
\chi^{\lambda/\nu}(\mu) &= \langle s_{\lambda/\nu}, p_\mu \rangle \\
&= \langle s_\lambda, p_\mu s_\nu \rangle && \text{(Theorem 4.5.3)} \\
&= \sum_T (-1)^{\mathrm{ht}(T)} && \text{(Corollary 4.2.9)}
\end{aligned}
$$

where the sum is the one we want.   □

Corollary 5.3.6. *Let $n = |\lambda/\mu|$. Then $\chi^{\lambda/\mu}(1^n) = f^{\lambda/\mu}$, the number of standard Young tableaux of shape λ/μ. In particular, $\chi^\lambda(1^{|\lambda|}) > 0$, so χ^λ is the character of the symmetric group $\mathfrak{S}_{|\lambda|}$ of an irreducible representation of dimension f^λ.*

Proof. By definition, a border-strip tableau of type (1^n) is the same as a standard Young tableau, and its height is always 0. The last statement follows from the discussion at the end of the previous section.   □

Corollary 5.3.7. *The Littlewood–Richardson coefficient $c_{\mu,\nu}^\lambda$ is non-negative.*

Proof. Recall that $s_\nu s_\mu = \sum_\lambda c_{\mu,\nu}^\lambda s_\lambda$. Applying ch^{-1}, this becomes $\chi^\nu \circ \chi^\mu = \sum_\lambda c_{\mu,\nu}^\lambda \chi^\lambda$. The induction of the character of a representation is again the character of a representation, so the right hand side is the character of a representation. Since every character is a non-negative sum of the irreducible ones, and the χ^λ are the irreducible characters, we conclude that $c_{\mu,\nu}^\lambda \geq 0$.   □

A natural followup: since $c_{\mu,\nu}^\lambda$ is a non-negative integer, is it the cardinality of some combinatorially meaningful set? We will give some constructions of such sets coming from tableaux later.

Finally, we match up the characters we found with the Specht modules.

Proposition 5.3.8. *χ^λ is the character of the Specht module $\mathbf{S}^\lambda$.*

Proof. We do this by descending induction on the set of partitions of a fixed size n under dominance order. The base case is $\lambda = (n)$. Since $h_n = \sum_\lambda z_\lambda^{-1} p_\lambda$ by Theorem 3.6.3, we see that $\chi^{(n)}(\mu) = 1$ for all μ and hence $\chi^{(n)}$ is the trivial character. We also know that $\mathbf{S}^{(n)}$ is the trivial representation, so these match up.

The base case implies that $\mathrm{ch}^{-1}(h_\lambda)$ is the character of $\mathbf{M}^\lambda \cong \mathrm{Ind}_{\mathfrak{S}_{\lambda_1} \times \cdots \mathfrak{S}_{\lambda_r}}^{\mathfrak{S}_n} \mathbf{C}$. Now suppose that we have proven the result for all partitions that dominate λ. By Lemma 2.3.10, if there is a Specht module $\mathbf{S}^\mu$ which is a direct summand of $\mathbf{M}^\lambda$, then $\mu \geq \lambda$. On the other hand, we have $h_\lambda = s_\lambda + \sum_{\mu > \lambda} K_{\mu,\lambda} s_\mu$. By the induction hypothesis, for any μ appearing in the sum, we have that χ^μ is the character of $\mathbf{S}^\mu$. Hence, χ^λ must be the character of some Specht module which is a summand of $\mathbf{M}^\lambda$ but is not isomorphic to $\mathbf{S}^\mu$ for any $\mu > \lambda$. But the only one satisfying this condition is $\mathbf{S}^\lambda$.   □

6. Combinatorial formulas

6.1. **Standard Young tableaux.** Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, recall that f^λ is the number of standard Young tableaux of shape λ. We gave a representation-theoretic interpretation in terms of symmetric groups: $f^\lambda = \chi^\lambda(1^n)$ is the dimension of an irreducible representation. Our goal now is to give some formulas for f^λ.

Theorem 6.1.1. *Pick $k \geq \ell(\lambda)$ and define $\ell_i = \lambda_i + k - i$. Then*

$$f^\lambda = \frac{n!}{\ell_1! \cdots \ell_k!} \prod_{1 \leq i < j \leq k} (\ell_i - \ell_j).$$

Proof. From §5.3, we have

$$p_\mu = \sum_\lambda \chi^\lambda(\mu) s_\lambda.$$

Now work in k variables $x_1, \ldots, x_k$. Recall from §4.4 that $s_\lambda = a_{\lambda+\rho}/a_\rho$. Multiply both sides above by a_ρ to get

$$a_\rho p_\mu = \sum_\lambda \chi^\lambda(\mu) a_{\lambda+\rho}.$$

If λ, λ' are partitions, the coefficient of $x^{\lambda+\rho}$ in $a_{\lambda'+\rho}$ is $\delta_{\lambda,\lambda'}$. In particular, we conclude that $\chi^\lambda(\mu)$ is the coefficient of $x^{\lambda+\rho}$ in $a_\rho p_\mu$. We're interested in the case $\mu = 1^n$.

First, by definition, we have

$$a_\rho = \det(x_i^{k-j})_{i,j=1}^k = \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) x_1^{k-\sigma(1)} \cdots x_k^{k-\sigma(k)}$$

and

$$p_{1^n} = (x_1 + x_2 + \cdots + x_k)^n = \sum_{i_1, \ldots, i_k} \binom{n}{i_1, \ldots, i_k} x_1^{i_1} \cdots x_k^{i_k}$$

where the sum is over all integers $i_1, \ldots, i_k \geq 0$ such that $i_1 + \cdots + i_k = n$ and

$$\binom{n}{i_1, \ldots, i_k} = \frac{n!}{i_1! \cdots i_k!}$$

is the multinomial coefficient. Hence the coefficient of $x^{\lambda+\rho}$ in $a_\rho p_{1^n}$ is

$$\sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) \binom{n}{\ell_1 - k + \sigma(1), \ldots, \ell_k - k + \sigma(k)} = \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) \frac{n!}{(\ell_1 - k + \sigma(1))! \cdots (\lambda_k - k + \sigma(k))!}$$

where, by convention, the sum is over σ such that the binomial coefficients make sense (i.e., $\ell_i + \sigma(i) \geq k$ for all i). Define $(x)_r = x(x-1)\cdots(x-r+1)$ so that $(\ell_i - k + \sigma(i))! = \ell_i!/(\ell_i)_{k-\sigma(i)}$. Then we can further rewrite this as

$$\frac{n!}{\ell_1! \cdots \ell_k!} \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) \prod_{i=1}^k (\ell_i)_{k-\sigma(i)} = \frac{n!}{\ell_1! \cdots \ell_k!} \det \begin{pmatrix} (\ell_1)_{k-1} & (\ell_1)_{k-2} & \cdots & (\ell_1)_2 & \ell_1 & 1 \\ (\ell_2)_{k-1} & (\ell_2)_{k-2} & \cdots & (\ell_2)_2 & \ell_2 & 1 \\ \vdots & & & & & \vdots \\ (\ell_k)_{k-1} & (\ell_k)_{k-2} & \cdots & (\ell_k)_2 & \ell_k & 1 \end{pmatrix}$$

This determinant is in fact equal to $\prod_{1 \leq i < j \leq k}(\ell_i - \ell_j)$. To see this, we can either use column operations and reduce it to the matrix $a_\rho(\ell_1, \ldots, \ell_k)$. Alternatively, replace the ℓ_i with variables x_i, and note that $(x_i - x_j)$ divides the determinant for all $i < j$ and that it has the same degree and leading coefficient as $\prod_{1 \leq i < j \leq k}(x_i - x_j)$. $\qquad\square$

We can deduce another nice combinatorial formula from this one using the notion of hook lengths. Given a box (i, j) in the Young diagram of λ, its hook is the set of boxes to the

right and below it (including itself). Its **hook length** $h(i,j)$ is the number of boxes in the book. Below, we list the hook lengths for the partition $(6,3,1)$:

8	6	5	3	2	1
4	2	1
1

Theorem 6.1.2 (Hook length formula). *If λ is a partition of n with Young diagram $Y(\lambda)$, then*

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in Y(\lambda)} h(i,j)}.$$

Example 6.1.3. Take $\lambda = (6,3,1)$. From the previous example, the hook length formula gives

$$f^{(6,3,1)} = \frac{10!}{8\cdot 6\cdot 5\cdot 3\cdot 2\cdot 4\cdot 2} = 315.$$

The formula in Theorem 6.1.1 gives

$$f^{(6,3,1)} = \frac{10!}{8!4!}(8-1)(8-4)(4-1) = 315. \qquad \square$$

Proof. Let $g^\lambda = \frac{n!}{\prod_{(i,j)\in Y(\lambda)} h(i,j)}$. We will show by induction on the number of columns of λ that $f^\lambda = g^\lambda$ using the formula $f^\lambda = \frac{n!}{\ell_1!\cdots\ell_k!}\prod_{1\le i<j\le k}(\ell_i - \ell_j)$ where $k = \ell(\lambda)$ and $\ell_i = \lambda_i + k - i$. If the number of columns is 1, then $g^\lambda = 1$ and $f^\lambda = 1$ by its definition as the number of standard Young tableaux.

In general, let μ be the partition obtained from λ by removing its first column. Note that the hook lengths in the first column of λ are $\ell_1, \ell_2, \ldots, \ell_k$, but otherwise, the hook lengths in the other boxes in λ are the hook lengths of the boxes in μ. Hence,

$$g^\mu = \frac{(n-k)!}{n!}\ell_1\cdots\ell_k g^\lambda.$$

On the other hand, f^μ and f^λ satisfy the same relation, so by induction, we conclude that $f^\lambda = g^\lambda$. $\qquad \square$

Remark 6.1.4. The hook length statistic appears in an ad hoc way in the above derivation. For a more natural derivation that uses the hook lengths in an essential way, see [GNW]. $\quad \square$

Remark 6.1.5. A natural question: given n, which partition λ maximizes f^λ? One might expect the partition closest to the staircase partition $(r, r-1, \ldots, 2, 1)$ to do this, but this isn't right. Note that $\sum_{\lambda \vdash n}(f^\lambda)^2 = n!$, so we put a probability measure on the partitions of n by choosing λ with probability $(f^\lambda)^2/n!$. Represent them by their Young diagram and normalize so that each box has area $1/n$. For the following, we will use the Russian convention. Work of Logan–Shepp and Vershik–Kerov show that there is a limiting curve for the boundary of our randomly chosen partition, and even give a formula for it:

$$\Omega(x) = \begin{cases} \frac{2}{\pi}(x\arcsin(\frac{x}{2}) + \sqrt{4-x^2}) & \text{if } |x| \le 2 \\ |x| & \text{if } |x| > 2 \end{cases}.$$

We plot Ω as the top curve below. The bottom portion is $|x|$ and represents the sides of other boundaries of the partition.
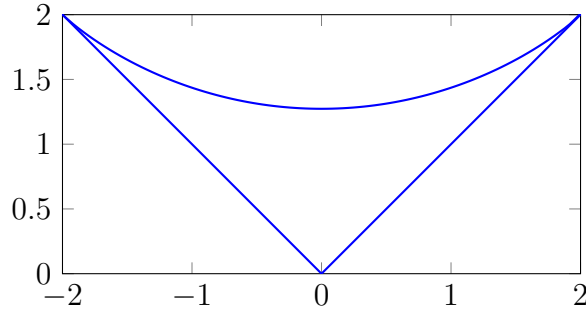
See [O] for a survey and further references. This also shows that the largest part of a random partition λ is $\lambda_1 \sim 2\sqrt{n}$ (and symmetrically, $\ell(\lambda) \sim 2\sqrt{n}$). $\qquad\square$

6.2. **Semistandard Young tableaux.** Now we derive a formula for the number of semistandard Young tableaux of shape λ using the numbers $1, \ldots, k$, i.e., for the evaluation $s_\lambda(1, 1, \ldots, 1)$ (k instances of 1). This is the dimension of an irreducible polynomial representation $\mathbf{S}_\lambda(\mathbf{C}^k)$ of $\mathbf{GL}_k(\mathbf{C})$.

Theorem 6.2.1.

$$\dim \mathbf{S}_\lambda(\mathbf{C}^k) = s_\lambda(1, \ldots, 1) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

where there are k instances of 1 above.

Proof. Work in finitely many variables $x_1, \ldots, x_k$ and use the determinantal formula in §4.4

$$s_\lambda(x_1, \ldots, x_k) = \frac{\det(x_i^{\lambda_j + k - j})_{i,j=1}^k}{\det(x_i^{k-j})_{i,j=1}^k}.$$

We can't evaluate $x_i = 1$ directly since we'd get $0/0$, but the following method let's us get around that. Let q be a new indeterminate and set $x_i = q^{i-1}$. Then

$$s_\lambda(1, q, \ldots, q^{k-1}) = \frac{\det(q^{i(\lambda_j + k - j)})_{i,j=1}^k}{\det(q^{i(k-j)})_{i,j=1}^k}.$$

Now in fact, both determinants become Vandermonde matrices, so we can simplify (see exercises):

$$s_\lambda(1, q, \ldots, q^{k-1}) = \frac{\prod_{1 \leq i < j \leq k}(q^{\lambda_i + k - i} - q^{\lambda_j + k - j})}{\prod_{1 \leq i < j \leq k}(q^{k-i} - q^{k-j})} = \prod_{1 \leq i < j \leq k} q^{\lambda_j} \frac{q^{\lambda_i - \lambda_j + j - i} - 1}{q^{j-i} - 1}.$$

Now we can set $q = 1$ in the final expression (either by dividing the polynomials, or using l'Hôpital's rule) and get the desired formula. $\qquad\square$

Corollary 6.2.2. *Keep k fixed. The function $n \mapsto \dim \mathbf{S}_{n\lambda}(\mathbf{C}^k)$ is a polynomial in n whose degree is the number of pairs $i < j$ such that $\lambda_i \neq \lambda_j$.*

Remark 6.2.3. (For those who know some algebraic geometry) The function $n \mapsto \dim \mathbf{S}_{n\lambda}(\mathbf{C}^k)$ is actually the Hilbert function of a projective embedding of a partial flag variety. More specifically, let $1 \leq i_1 < i_2 < \cdots < i_r$ be the indices such that $\lambda_{i_j} \neq \lambda_{i_j+1}$. Then the collection of subspaces $F_1 \subset \cdots \subset F_r \subset \mathbf{C}^k$ where $\dim F_j = i_j$ has the structure of a projective algebraic variety which admits an embedding into the projective space on $\mathbf{S}_\lambda(\mathbf{C}^k)$ giving rise to the Hilbert function we're talking about. $\qquad\square$

Given a box $(i,j) \in Y(\lambda)$, define its **content** to be $c(i,j) = j - i$.

Theorem 6.2.4 (Hook-content formula)**.**

$$\dim \mathbf{S}_\lambda(\mathbf{C}^k) = s_\lambda(1,\ldots,1) = \prod_{(i,j) \in Y(\lambda)} \frac{k + c(i,j)}{h(i,j)}$$

where there are k instances of 1 *above.*

Proof. Let $n = |\lambda|$ and set $\ell_i = \lambda_i + k - i$. Using the formulas for f^λ in the previous section, we have

$$\prod_{(i,j) \in Y(\lambda)} \frac{k + c(i,j)}{h(i,j)} = \frac{f^\lambda}{n!} \prod_{(i,j) \in Y(\lambda)} (k - i + j) = \frac{1}{\ell_1! \cdots \ell_k!} \prod_{1 \le i < j \le k} (\ell_i - \ell_j) \prod_{(i,j) \in Y(\lambda)} (k - i + j).$$

Next, note that

$$\prod_{(i,j) \in Y(\lambda)} (k - i + j) = \prod_{i=1}^{k} \frac{\ell_i!}{(k - i)!},$$

so the above simplifies to

$$\frac{\prod_{1 \le i < j \le k} (\ell_i - \ell_j)}{(k-1)!(k-2)! \cdots 2!} = \prod_{1 \le i < j \le k} \frac{\lambda_i - \lambda_j - i + j}{j - i},$$

and the latter we have shown to be $\dim \mathbf{S}_\lambda(\mathbf{C}^k)$. $\square$

Corollary 6.2.5. *For each partition λ, the function $n \mapsto \dim \mathbf{S}_\lambda(\mathbf{C}^n)$ is a polynomial in n.*

The Jacobi–Trudi identity (Theorem 4.6.1) gives yet another formula. It is easy to see that $h_n(1,\ldots,1) = \binom{n+k-1}{n}$ (k 1's here, and this is the number of monomials of degree n with k variables).

Theorem 6.2.6. *If $n \ge \ell(\lambda)$, then*

$$\dim \mathbf{S}_\lambda(\mathbf{C}^k) = s_\lambda(1,\ldots,1) = \det\left(\binom{k + \lambda_i - i + j - 1}{k - 1}\right)_{i,j=1}^{n}.$$

6.3. **Littlewood–Richardson coefficients.** We have encountered Littlewood–Richardson coefficients $c_{\lambda,\mu}^\nu$ in several contexts now:

- The multiplication of Schur functions: $s_\lambda s_\mu = \sum_\nu c_{\lambda,\mu}^\nu s_\nu$,
- The expansion of a skew Schur function: $s_{\nu/\mu} = \sum_\lambda c_{\lambda,\mu}^\nu s_\lambda$,
- The induction of symmetric group characters: $\chi^\lambda \circ \chi^\mu = \sum_\nu c_{\lambda,\mu}^\nu \chi^\nu$,
- The restriction of a symmetric group character: $\mathrm{Res}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} \chi^\nu = \sum_{\lambda,\mu} c_{\lambda,\mu}^\nu (\chi^\lambda \otimes \chi^\mu)$,

We'll see another instance in the next section. Here we'll give one way to compute $c_{\lambda,\mu}^\nu$ (without proof). See [Sta, §7, Appendix 1] for details and more formulas.

Let $w = w_1 w_2 \cdots w_n$ be a sequence of positive integers and let $m_i(w)$ be the number of w_j equal to i. A prefix of w is any subsequence of the form $w_1 w_2 \cdots w_m$ for $m \le n$. We say that w is a **lattice permutation** (also called **Yamanouchi word** or **ballot sequence**) if, for every prefix v of w, we have $m_i(v) \ge m_{i+1}(v)$ for all i. Given a tableau T, its **reverse reading word** is the list of the entries of T in the following order: start with row 1 and list the entries from right to left, move to row 2 and list the entries from right to left, etc.

Call a Littlewood–Richardson tableau a SSYT of skew shape whose reverse reading word is a lattice permutation.

Theorem 6.3.1. $c_{\lambda,\mu}^{\nu}$ *is the number of Littlewood–Richardson tableaux of shape* ν/μ *and type* λ.

Recall that $c_{\lambda,\mu}^{\nu} = c_{\mu,\lambda}^{\nu}$, so there is a big asymmetry in this description for $c_{\lambda,\mu}^{\nu}$. This can sometimes be a good thing: one set of SSYT may be much easier to describe than the other, even though they must have the same size. It is also possible to give descriptions that are symmetric in μ and λ, but we will not discuss that here.

Example 6.3.2. Let $\lambda = (4, 2, 1)$, $\mu = (5, 2)$, $\nu = (6, 5, 2, 1)$. Then $c_{\lambda,\mu}^{\nu} = 3$; here are all of the SSYT of shape ν/μ of type λ whose reverse reading words are lattice permutations together with their reverse reading words:
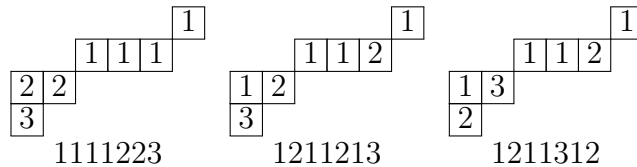
Alternatively, we could count the number of SSYT of shape ν/λ of type μ whose reverse reading words are lattice permutations. We list them here:
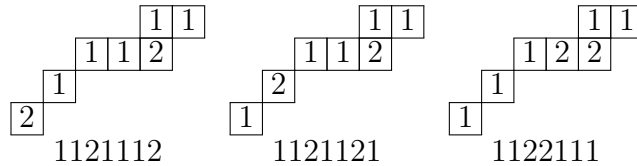
Remark 6.3.3. If $\lambda = (d)$, then $c_{\lambda,\mu}^{\nu}$ can be computed by the Pieri rule. In fact, the description given above directly generalizes this: a SSYT of shape ν/μ of type (d) cannot have more than one box in a single column. But given any collection of boxes, no two in a single column, putting a 1 in each box gives a valid SSYT whose reverse reading word is a lattice permutation. So we conclude that it's 1 if ν/μ is a horizontal strip and 0 otherwise.

Similarly, if $\lambda = (1^d)$, then consider a SSYT of shape ν/μ of type (1^d) whose reverse reading word is a lattice permutation. The reverse reading word must then be $123 \cdots d$. But since it's a SSYT, no two of these entries can appear in the same row. So we conclude that it's 1 if ν/μ is a vertical strip and 0 otherwise. □

7. Polynomial functors

Let $\mathbf{k}$ be a field. We have already seen the construction of the tensor product of two vector spaces $V \otimes W$. Here we give a few related constructions. We let $\mathbf{GL}(V)$ be the group of invertible linear operators on V. This is the **general linear group**. This is an infinite group if $\mathbf{k}$ is an infinite field, and we will study a special class of its representations. The effect of applying elements of $\mathbf{GL}(V)$ to V is to do a change of basis, and we will construct new representations out of V using "natural" constructions. Heuristically, natural means that it does not depend on a choice of basis for V.

7.1. **Basic multilinear algebra.** For a positive integer d, let $V^{\otimes d} = V \otimes \cdots \otimes V$ where the right hand side contains d instances of V. This space carries a (right) action of the symmetric group $\mathfrak{S}_d$:

$$\sigma \cdot \left(\sum v_1 \otimes \cdots \otimes v_d \right) = \sum v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

It also carries a (left) action of $\mathbf{GL}(V)$:

$$g \cdot \left(\sum v_1 \otimes \cdots \otimes v_d \right) = \sum (g \cdot v_1) \otimes \cdots \otimes (g \cdot v_d).$$

These actions commute with one another, meaning that $g \cdot (\sigma \cdot \mathbf{v}) = \sigma \cdot (g \cdot \mathbf{v})$ for any $\mathbf{v} \in V^{\otimes d}$, $g \in \mathbf{GL}(V)$, and $\sigma \in \mathfrak{S}_d$.

We define the d**th symmetric power** $\mathrm{Sym}^d V$ of V to be the quotient of $V^{\otimes d}$ by the subspace spanned by elements of the form $\mathbf{v} - \sigma \cdot \mathbf{v}$ where $\mathbf{v} \in V^{\otimes d}$ and $\sigma \in \mathfrak{S}_d$. The fact that the actions of $\mathbf{GL}(V)$ and $\mathfrak{S}_d$ commute on $V^{\otimes d}$ implies that this subspace is a $\mathbf{GL}(V)$-subrepresentation, and hence $\mathrm{Sym}^d V$ is a $\mathbf{GL}(V)$-representation as well.

Given an element $v_1 \otimes \cdots \otimes v_d$, we let $v_1 \cdots v_d$ denote its image in $\mathrm{Sym}^d V$. For $g \in \mathbf{GL}(V)$, the action is given by $g \cdot (v_1 \cdots v_d) = (g \cdot v_1) \cdots (g \cdot v_d)$. If $e_1, \ldots, e_n$ is a basis for V, then recall that $\{e_{i_1} \otimes \cdots \otimes e_{i_d} \mid 1 \le i_j \le n\}$ is a basis for $V^{\otimes d}$. Then it is easy to verify that $\{e_{i_1} \cdots e_{i_d} \mid 1 \le i_1 \le i_2 \le \cdots \le i_d \le n\}$ is a basis for $\mathrm{Sym}^d V$, so that its dimension is $\binom{n+d-1}{d}$.

There is a natural "multiplication" map

$$\mu \colon \mathrm{Sym}^d V \otimes \mathrm{Sym}^e V \to \mathrm{Sym}^{d+e} V$$
$$(v_1 \cdots v_d) \otimes (w_1 \cdots w_e) \mapsto v_1 \cdots v_d w_1 \cdots w_e$$

which is associative in the appropriate sense (note: we are showing the effect on basic tensors, and for general elements, we must use linearity, i.e., distribute). Most importantly, the definition of μ does not depend on a choice of basis for V. This means that μ is a $\mathbf{GL}(V)$-equivariant map.

The d**th exterior power** $\bigwedge^d V$ of V is the quotient of $V^{\otimes d}$ by the subspace spanned by elements of the form $v_1 \otimes \cdots \otimes v_d$ where $v_i = v_j$ for some $i \ne j$. As before, $\bigwedge^d V$ is a $\mathbf{GL}(V)$-representation. We let $v_1 \wedge \cdots \wedge v_d$ denote the image of $v_1 \otimes \cdots \otimes v_d$ in $\bigwedge^d V$. The action is given by $g \cdot (v_1 \wedge \cdots \wedge v_d) = (g \cdot v_1) \wedge \cdots \wedge (g \cdot v_d)$. Note that swapping two elements introduces a sign: $v_1 \wedge v_2 = -v_2 \wedge v_1$ since

$$0 = (v_1 + v_2) \wedge (v_1 + v_2) = v_1 \wedge v_1 + v_1 \wedge v_2 + v_2 \wedge v_1 + v_2 \wedge v_2 = v_1 \wedge v_2 + v_2 \wedge v_1.$$

Then $\{e_{i_1} \wedge \cdots \wedge e_{i_d} \mid 1 \le i_1 < \cdots < i_d \le n\}$ is a basis for $\bigwedge^d V$, so that its dimension is $\binom{n}{d}$.

There is a natural "multiplication" map

$$\mu \colon \bigwedge^d V \otimes \bigwedge^e V \to \bigwedge^{d+e} V$$
$$(v_1 \wedge \cdots \wedge v_d) \otimes (w_1 \wedge \cdots \wedge w_e) \mapsto v_1 \wedge \cdots \wedge v_d \wedge w_1 \wedge \cdots \wedge w_e$$

which is again associative in the appropriate sense. Again, μ is $\mathbf{GL}(V)$-equivariant. We will also need something dual to multiplication called comultiplication:

$$\Delta\colon \bigwedge^{d+e} V \to \bigwedge^{d} V \otimes \bigwedge^{e} V$$

$$v_1 \wedge \cdots \wedge v_{d+e} \mapsto \sum_I \operatorname{sgn}(I, I^c) v_I \otimes v_{I^c}$$

where the notation is as follows. The sum is over all d-element subsets I of $[d+e] = \{1, \ldots, d+e\}$, and I^c is the complement of I. The notation v_I means $v_{i_1} \wedge \cdots \wedge v_{i_d}$ where $i_1 < i_2 < \cdots < i_d$ are the elements of I, and similarly we define v_{I^c}. Finally, $\operatorname{sgn}(I, I^c) = \pm 1$ and is determined by the equation

$$\operatorname{sgn}(I, I^c) v_I \wedge v_{I^c} = v_1 \wedge \cdots \wedge v_{d+e}.$$

We leave it to an exercise to verify that Δ is $\mathbf{GL}(V)$-equivariant. Note that there are two different ways to get from $\bigwedge^{d+e+f} V$ to $\bigwedge^d V \otimes \bigwedge^e V \otimes \bigwedge^f V$ using comultiplication twice. In fact, they are the same, which we will express by saying that Δ is coassociative. We can also apply this $d-1$ times to get a map $\bigwedge^d V \to V^{\otimes d}$, with the property that the composition

$$\bigwedge^{d} V \to V^{\otimes d} \to \bigwedge^{d} V$$

(where the second map is the quotient map coming from the definition of $\bigwedge^d V$) is $d!$ times the identity. In particular, if the characteristic of $\mathbf{k}$ is 0 or larger than d, then $\bigwedge^d V$ is a direct summand of $\bigwedge^{\otimes d}$ as $\mathbf{GL}(V)$-representations.

There is also a comultiplication map for symmetric powers, but we will not need it.

Finally, we can extend everything above to $d = 0$ by declaring that $V^{\otimes 0}$ is the trivial vector space $\mathbf{k}$.

7.2. **Schur functors.** We will now use exterior and symmetric powers to build a more complicated set of natural operations, depending on integer partitions.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of d and set $\mu = \lambda^\dagger = (\mu_1, \ldots, \mu_s)$. The **Schur functor** $\mathbf{S}_\lambda V$ is defined to be the image of the following composition:

$$\bigwedge^{\mu_1} V \otimes \cdots \otimes \bigwedge^{\mu_s} V \to V^{\otimes \mu_1} \otimes \cdots \otimes V^{\otimes \mu_s}$$
$$\to V^{\otimes \lambda_1} \otimes \cdots \otimes V^{\otimes \lambda_r}$$
$$\to \operatorname{Sym}^{\lambda_1} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_r} V,$$

where the first map is given by comultiplication, the second map is a certain reordering that we will explain shortly, and the third map is multiplication.

The reordering is best understood in terms of Young diagrams, which we will illustrate with an example.

Example 7.2.1. Consider $\lambda = (3, 2)$ so that $\mu = (2, 2, 1)$. Then $\bigwedge^2 V \otimes \bigwedge^2 V \otimes V$ is spanned by elements of the form $(v_1 \wedge v_2) \otimes (v_3 \wedge v_4) \otimes v_5$. We can record element of $V^{\otimes 5}$ by putting vectors into the boxes of $Y(\lambda)$. The order we do this in depends: if we write $V^{\otimes 5} = V^{\otimes \mu_1} \otimes \cdots \otimes V^{\otimes \mu_s}$, then we will think of these as the columns of $Y(\lambda)$. On the other

hand, writing $V^{\otimes 5} = V^{\otimes \lambda_1} \otimes \cdots \otimes V^{\otimes \lambda_r}$, we will instead think of these as the rows of the $Y(\lambda)$. The map then looks as follows:

$$(v_1 \wedge v_2) \otimes (v_3 \wedge v_4) \otimes v_5 \mapsto \begin{array}{|c|c|c|} \hline v_1 & v_3 & v_5 \\ \hline v_2 & v_4 \\ \cline{1-2} \end{array} - \begin{array}{|c|c|c|} \hline v_2 & v_3 & v_5 \\ \hline v_1 & v_4 \\ \cline{1-2} \end{array} - \begin{array}{|c|c|c|} \hline v_1 & v_4 & v_5 \\ \hline v_2 & v_3 \\ \cline{1-2} \end{array} + \begin{array}{|c|c|c|} \hline v_2 & v_4 & v_5 \\ \hline v_1 & v_3 \\ \cline{1-2} \end{array}$$

$$\mapsto v_1 v_3 v_5 \otimes v_2 v_4 - v_2 v_3 v_5 \otimes v_1 v_4 - v_1 v_4 v_5 \otimes v_2 v_3 + v_2 v_4 v_5 \otimes v_1 v_3. \quad \square$$

Since all of the maps are $\mathbf{GL}(V)$-equivariant, we see that $\mathbf{S}_\lambda(V)$ is a $\mathbf{GL}(V)$-representation. It follows immediately from the definition that $\mathbf{S}_\lambda(V) = 0$ if $\ell(\lambda) > \dim V$ since the corresponding exterior power $\bigwedge^{\mu_1} V$ is 0.

Example 7.2.2. There are two extreme cases that we already know. If $\lambda = (d)$, then the map becomes the quotient map $V^{\otimes d} \to \operatorname{Sym}^d V$ so that $\mathbf{S}_{(d)} V = \operatorname{Sym}^d V$. On the other hand, if $\lambda = (1^d)$, then the map becomes the comultiplication map $\bigwedge^d V \to V^{\otimes d}$, which is injective, so $\mathbf{S}_{(1^d)} V = \bigwedge^d V$. $\hspace{2cm} \square$

Fix a basis $e_1, \ldots, e_n$ for V. We would like to find a basis for $\mathbf{S}_\lambda V$. Given a tableau T on $Y(\lambda)$, we get a vector in $\bigwedge^{\mu_1} V \otimes \cdots \otimes \bigwedge^{\mu_s} V$ by taking

$$(e_{T_{1,1}} \wedge e_{T_{2,1}} \wedge \cdots \wedge e_{T_{\mu_1,1}}) \otimes \cdots \otimes (e_{T_{1,s}} \wedge \cdots \wedge e_{T_{\mu_s,s}});$$

let e_T be its image in $\mathbf{S}_\lambda V$. Recall that T is semistandard if $T_{i,j} \le T_{i,j+1}$ and $T_{i,j} < T_{i+1,j}$ for all i, j where that makes sense.

Theorem 7.2.3. $\{e_T \mid T \text{ is semistandard}\}$ is a basis for $\mathbf{S}_\lambda V$.

We will omit the proof.

Note that this recovers the bases we described for both $\operatorname{Sym}^d V$ and $\bigwedge^d V$.

7.3. **Polynomial representations and characters.** Let $\mathbf{GL}_n(\mathbf{C})$ denote the group of invertible $n \times n$ complex matrices.

A **polynomial representation** of $\mathbf{GL}_n(\mathbf{C})$ is a homomorphism $\rho \colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V)$ where V is a $\mathbf{C}$-vector space, and the entries of ρ can be expressed in terms of polynomials (as soon as we pick a basis for V).

A simple example is the identity map $\rho \colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}_n(\mathbf{C})$. Slightly more sophisticated is $\rho \colon \mathbf{GL}_2(\mathbf{C}) \to \mathbf{GL}(\operatorname{Sym}^2(\mathbf{C}^2))$. Pick a basis $\{x, y\}$ for $\mathbf{C}^2$. The homomorphism can be defined by linear change of coordinates, i.e.,

$$\rho(g)(ax^2 + bxy + cy^2) = a(gx)^2 + b(gx)(gy) + c(gy)^2.$$

If we pick the basis x^2, xy, y^2 for $\operatorname{Sym}^2(\mathbf{C}^2)$, this can be written in coordinates as

$$\mathbf{GL}_2(\mathbf{C}) \to \mathbf{GL}_3(\mathbf{C})$$

$$(7.3.1) \qquad \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} \mapsto \begin{pmatrix} g_{1,1}^2 & g_{1,1}g_{1,2} & g_{1,2}^2 \\ 2g_{1,1}g_{2,1} & g_{1,1}g_{2,2} + g_{1,2}g_{2,1} & 2g_{1,2}g_{2,2} \\ g_{2,1}^2 & g_{2,1}g_{2,2} & g_{2,2}^2 \end{pmatrix}.$$

More generally, we can define $\rho \colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(\operatorname{Sym}^d(\mathbf{C}^n))$ for any n, d. Another important example uses exterior powers instead of symmetric powers, so we have $\rho \colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(\wedge^d(\mathbf{C}^n))$. From the definition, we see that the property of being polynomial is preserved by taking tensor products, direct sums, subrepresentations, and quotient representations. So we immediately see that every Schur functor $\mathbf{S}_\lambda(\mathbf{C}^n)$ defines a polynomial representation for $\mathbf{GL}_n(\mathbf{C})$.

An important invariant of a polynomial representation ρ is its **character**: define
$$\text{char}(\rho)(x_1,\ldots,x_n) := \text{Tr}(\rho(\text{diag}(x_1,\ldots,x_n))),$$
where $\text{diag}(x_1,\ldots,x_n)$ is the diagonal matrix with entries $x_1,\ldots,x_n$ and Tr denotes trace.

Lemma 7.3.2. $\text{char}(\rho)(x_1,\ldots,x_n) \in \Lambda(n)$.

Proof. Each permutation $\sigma \in \mathfrak{S}_n$ corresponds to a permutation matrix $M(\sigma)$: this is the matrix with a 1 in row $\sigma(i)$ and column i for $i = 1,\ldots,n$ and 0's everywhere else. Then
$$M(\sigma)^{-1}\text{diag}(x_1,\ldots,x_n)M(\sigma) = \text{diag}(x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$
Now use that the trace of a matrix is invariant under conjugation:
$$\begin{aligned}
\text{char}(\rho)(x_1,\ldots,x_n) &= \text{Tr}(\rho(\text{diag}(x_1,\ldots,x_n)))\\
&= \text{Tr}(\rho(M(\sigma))^{-1}\rho(\text{diag}(x_1,\ldots,x_n))\rho(M(\sigma)))\\
&= \text{Tr}(\rho(M(\sigma)^{-1}\text{diag}(x_1,\ldots,x_n)M(\sigma)))\\
&= \text{Tr}(\rho(\text{diag}(x_{\sigma(1)},\ldots,x_{\sigma(n)})))\\
&= \text{char}(\rho)(x_{\sigma(1)},\ldots,x_{\sigma(n)}). \qquad\square
\end{aligned}$$

Example 7.3.3.
- The character of the identity representation is $x_1 + x_2 + \cdots + x_n$.
- The character of the representation $\rho\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(\text{Sym}^d(\mathbf{C}^n))$ is
$$h_d(x_1,\ldots,x_n) = \sum_{1 \le i_1 \le \cdots \le i_d \le n} x_{i_1}\cdots x_{i_d}.$$
- The character of the representation $\rho\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(\wedge^d(\mathbf{C}^n))$ is
$$e_d(x_1,\ldots,x_n) = \sum_{1 \le i_1 < \cdots < i_d \le n} x_{i_1}\cdots x_{i_d}.$$
- More generally, the character of $\rho\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(\mathbf{S}_\lambda(\mathbf{C}^n))$ is
$$s_\lambda(x_1,\ldots,x_n) = \sum_{T \text{ semistandard of shape } \lambda} x^T. \qquad\square$$

Basic operations transform easily on the level of characters:
- If $\rho_i\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V_i)$ are polynomial representations for $i = 1, 2$, we can form the direct sum representation $\rho_1 \oplus \rho_2\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V_1 \oplus V_2)$ via
$$(\rho_1 \oplus \rho_2)(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$
and
$$\text{char}(\rho_1 \oplus \rho_2)(x_1,\ldots,x_n) = \text{char}(\rho_1)(x_1,\ldots,x_n) + \text{char}(\rho_2)(x_1,\ldots,x_n).$$
- If $\rho_i\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V_i)$ are polynomial representations for $i = 1, 2$, we can form the tensor product representation $\rho_1 \otimes \rho_2\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V_1 \otimes V_2)$ via (assuming $\rho_1(g)$ is $N \times N$):
$$(\rho_1 \otimes \rho_2)(g) = \begin{pmatrix} \rho_1(g)_{1,1}\rho_2(g) & \rho_1(g)_{1,2}\rho_2(g) & \cdots & \rho_1(g)_{1,N}\rho_2(g) \\ \rho_1(g)_{2,1}\rho_2(g) & \rho_1(g)_{2,2}\rho_2(g) & \cdots & \rho_1(g)_{2,N}\rho_2(g) \\ \vdots & & & \\ \rho_1(g)_{N,1}\rho_2(g) & \rho_1(g)_{N,2}\rho_2(g) & \cdots & \rho_1(g)_{N,N}\rho_2(g) \end{pmatrix}$$

(here we are multiplying $\rho_2(g)$ by each entry of $\rho_1(g)$ and creating a giant block matrix) and

$$\operatorname{char}(\rho_1 \otimes \rho_2)(x_1, \ldots, x_n) = \operatorname{char}(\rho_1)(x_1, \ldots, x_n) \cdot \operatorname{char}(\rho_2)(x_1, \ldots, x_n).$$

7.4. **Re-interpreting symmetric function identities.** Here is a summary of some important facts whose proofs we will not have time to discuss (see a course on Lie algebras/groups, for example):

Theorem 7.4.1.   *(1) Finite-dimensional polynomial representations of $\mathbf{GL}_n(\mathbf{C})$ are semisimple, i.e., are isomorphic to a direct sum of simple polynomial representations.*
    (2) Two polynomial representations of $\mathbf{GL}_n(\mathbf{C})$ are isomorphic if and only if they have the same character.
    (3) The Schur functors $\mathbf{S}_\lambda(\mathbf{C}^n)$ for $n \geq \ell(\lambda)$ are irreducible and pairwise non-isomorphic. Every irreducible polynomial representation of $\mathbf{GL}_n(\mathbf{C}^n)$ is isomorphic to one of them.

With these facts in hand, we can now interpret properties and identities of symmetric functions into representation theory.

7.4.1. *Cauchy identities.* In finitely many variables $x_1, \ldots, x_n, y_1, \ldots, y_m$, $\prod_{i,j}(1 - x_i y_j)^{-1}$ is the character of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$ acting on $\operatorname{Sym}(\mathbf{C}^n \otimes \mathbf{C}^m) = \bigoplus_{d \geq 0} \operatorname{Sym}^d(\mathbf{C}^n \otimes \mathbf{C}^m)$. The Cauchy identity gives a decomposition into Schur functors:

$$\operatorname{Sym}(\mathbf{C}^n \otimes \mathbf{C}^m) \cong \bigoplus_\lambda \mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_\lambda(\mathbf{C}^m)$$

where the sum is over all partitions (or just those with $\ell(\lambda) \leq \min(m, n)$).

Similarly, $\prod_{i,j}(1 + x_i y_j)$ is the character of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$ acting on the exterior algebra $\bigwedge^\bullet(\mathbf{C}^n \otimes \mathbf{C}^m) = \bigoplus_{d \geq 0} \bigwedge^d(\mathbf{C}^n \otimes \mathbf{C}^m)$. The dual Cauchy identity gives a decomposition into Schur functors:

$$\bigwedge^\bullet(\mathbf{C}^n \otimes \mathbf{C}^m) \cong \bigoplus_\lambda \mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_{\lambda^\dagger}(\mathbf{C}^m),$$

where the sum is over all partitions (or just those with $\ell(\lambda) \leq n$ and $\lambda_1 \leq m$).

7.4.2. *Pieri and Littlewood–Richardson.* From the interpretation of s_λ as the character of an irreducible representation $\mathbf{S}_\lambda$, and the fact that polynomial representations are direct sums of irreducible ones, we can reinterpret the Littlewood–Richardson coefficient as the multiplicity of $\mathbf{S}_\lambda$ in the decomposition of the tensor product of $\mathbf{S}_\mu \otimes \mathbf{S}_\nu$. From this, it is immediate that $c^\lambda_{\mu,\nu} \geq 0$.

The Pieri rule describes the decomposition of the tensor product of $\mathbf{S}_\nu$ with an exterior power $\bigwedge^k$, respectively, a symmetric power Sym^k.

The decomposition of s_1^n can be interpreted as a decomposition of the tensor power of a vector space

$$(\mathbf{C}^d)^{\otimes n} = \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} \mathbf{S}_\lambda(\mathbf{C}^d)^{\oplus f^\lambda}.$$

Hence the multiplicity space of $\mathbf{S}_\lambda(\mathbf{C}^d)$ has dimension f^λ.

Remark 7.4.2 (Schur–Weyl duality). As in the beginning of the section, the space $(\mathbf{C}^n)^{\otimes d}$ has commuting actions of $\mathbf{GL}_n(\mathbf{C})$ and $\mathfrak{S}_d$. In particular, we can think of this as an action of the direct product $\mathbf{GL}_n(\mathbf{C}) \times \mathfrak{S}_d$.

Since this representation is semisimple for both groups, we can decompose this space as a sum of irreducible representations of $\mathbf{GL}_n(\mathbf{C}) \times \mathfrak{S}_d$. In fact, we get the following:

$$(\mathbf{C}^n)^{\otimes d} = \bigoplus_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq n}} \mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}^\lambda,$$

where $\mathbf{S}_\lambda(\mathbf{C}^n)$ is the Schur functor and $\mathbf{S}^\lambda$ is the Specht module of $\mathfrak{S}_d$. We will not prove this. $\square$

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Department of Mathematics, University of California, San Diego
URL: http://math.ucsd.edu/~ssam/
Email address: ssam@ucsd.edu