Math 188, Spring 2021
Homework 2
Due: April 16, 2021 11:59PM via Gradescope

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) (a) Let $(A_i(x))_{i\geq 0}$ be a sequence of formal power series such that $\lim_{i\to\infty} A_i(x) = A(x)$. Let σ be a permutation of the non-negative integers.
Prove or find a counterexample: $\lim_{i\to\infty} A_{\sigma(i)}(x)$ converges to $A(x)$.

(b) Suppose we are given a doubly-indexed sequence of formal power series $(A_{i,j}(x))_{i,j\geq 0}$ such that:

- For each i, the limit $\lim_{j\to\infty} A_{i,j}(x)$ converges to $B_i(x)$ and $\lim_{i\to\infty} B_i(x)$ converges to $B(x)$, and
- For each j, the limit $\lim_{i\to\infty} A_{i,j}(x)$ converges to $C_j(x)$ and $\lim_{j\to\infty} C_j(x)$ converges to $C(x)$.

Prove or find a counterexample: $B(x) = C(x)$

(2) Let $F(x)$ be a formal power series with $F(0) = 0$.

(a) Show that there exists a formal power series $G(x)$ with $G(0) = 0$ such that $F(G(x)) = x$ if and only if $[x^1]F(x) \neq 0$.

(b) Assuming $[x^1]F(x) \neq 0$, show that $G(x)$ is unique and also satisfies $G(F(x)) = x$. You may use without proof that composition of formal power series is associative.

(3) Assume now that we deal with complex-coefficient formal power series. Define the following sets of formal power series:

$$V = \{F(x) \mid F(0) = 0\},$$
$$W = \{G(x) \mid G(0) = 1\}.$$

(a) Given $F \in V$, show that $\mathbf{R}(F) = \sum_{n\geq 0} \frac{F(x)^n}{n!}$ is the *unique* formal power series $G \in W$ such that $DG(x) = DF(x)G(x)$. This defines a function $\mathbf{R}\colon V \to W$.
[Conventions: $F(x)^0 = 1$ even if $F(x) = 0$ and $0! = 1$]

(b) Given $G \in W$, show that there is a *unique* formal power series $F \in V$ such that $DF(x) = DG(x)/G(x)$. We define the function $\mathbf{L}\colon W \to V$ by $\mathbf{L}(G) = F$.
[For the rest, it is unnecessary to use explicit formulas for $\mathbf{L}$ and $\mathbf{R}$ and in fact it may be easier to only use the uniqueness properties above.]

(c) Show that $\mathbf{R}$ and $\mathbf{L}$ are inverses of each other.

(d) Show that $\mathbf{R}(F_1 + F_2) = \mathbf{R}(F_1)\mathbf{R}(F_2)$ for all $F_1, F_2 \in V$.

(e) Show that $\mathbf{L}(G_1G_2) = \mathbf{L}(G_1) + \mathbf{L}(G_2)$ for all $G_1, G_2 \in W$.

(f) If m is a positive integer and $G \in W$, show that $\mathbf{R}(\frac{\mathbf{L}G}{m})$ is an mth root of G. (Hence this gives an alternative proof for the existence of mth roots.)

(4) Let $n \geq 2$ be an integer.

(a) Prove that

$$\sum_{i=0}^{n} i\binom{n}{i}(-1)^{i-1} = 0.$$

(b) Compute

$$\sum_{\substack{0\leq i\leq n \\ i \text{ even}}} i\binom{n}{i}.$$

1. Optional problems (don't turn in)

(5) Let $A_0(x), A_1(x), \ldots$ and $B_0(x), B_1(x), \ldots$ be sequences of formal power series. Assume that $\lim_{i\to\infty} A_i(x) = A(x)$ and $\lim_{i\to\infty} B_i(x) = B(x)$. Prove that

$$\lim_{i\to\infty}(A_i(x) + B_i(x)) = A(x) + B(x),$$
$$\lim_{i\to\infty}(A_i(x)B_i(x)) = A(x)B(x).$$

(6) Which of the following infinite products of formal power series converge?
 (a) $\prod_{i\geq 0}(1 + x^{i+1})$
 (b) $\prod_{i\geq 0}(1 + x)^{i+1}$

(7) Give proofs for the following identities for formal derivatives of formal power series (for the second identity, assume $A(0) = 0$ so that the compositions are defined):

$$D(A \cdot B) = (DA) \cdot B + A \cdot (DB),$$
$$D(B \circ A) = (DA) \cdot (DB \circ A).$$

(8) Let $A(x)$ be a formal power series. Define its **order** $\mathrm{ord}(A)$ to be the smallest k such that $[x^k]A(x) \neq 0$. Show that there exists a formal power series $B(x)$ such that $B(x)^2 = A(x)$ if and only if the order k of $A(x)$ is even, and our field of scalars contains a square root of $[x^k]A(x)$.

(9) Use the notation from (3).
 (a) Show that if $\sum_{i\geq 0} F_i(x)$ converges to $F(x)$, then $\prod_{i\geq 0} \mathbf{R}(F_i)$ converges to $\mathbf{R}(F)$.
 (b) Show that if $\prod_{i\geq 0} G_i(x)$ converges to $G(x)$, then $\sum_{i\geq 0} \mathbf{R}(G_i)$ converges to $\mathbf{R}(G)$.

(10) I am not sure how easy the following is to prove without knowing the right trick, but I'm curious if you can come up with a nice solution.

Suppose we pick a sequence of numbers $(a_2, a_4, a_6, \ldots)$. Show that there is a unique way to complete it to a sequence $(a_n)_{n\geq 2}$ such that $A(x) = -x + \sum_{n\geq 2} a_n x^n$ satisfies $A(A(x)) = x$.