Math 188, Spring 2021
Homework 5
Due: May 7, 2021 11:59PM via Gradescope

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Prove the following identities about the number of integer partitions:
   (a) For $n \geq k$, $p_k(n) = p_{\leq k}(n-k)$.
   (b) For $n > 0$, the number of partitions of n not using 1 as a part is $p(n) - p(n-1)$.

(2) (a) Use the following q-analogue of Pascal's identity

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \qquad \text{(for } n \geq k > 0)$$

   to show that if d is a non-negative integer, then

$$\sum_{n \geq 0} \begin{bmatrix} n+d \\ n \end{bmatrix}_q x^n = \prod_{i=0}^{d} (1 - q^i x)^{-1} = \frac{1}{(1-x)(1-qx)\cdots(1-q^d x)}.$$

   (b) Give a direct explanation for why the coefficient of x^n of the right side is the sum $\sum_\lambda q^{|\lambda|}$ over all integer partitions λ fitting in the $n \times d$ rectangle.

(3) Let T be a subset of the positive integers such that if $x \in T$, then $2x \in T$. Define $2T = \{2x \mid x \in T\}$ and let $S = T \setminus 2T$.

   Let $p_S(n)$ be the number of partitions of n using only parts from S, and let $p_{T,\text{dist}}(n)$ be the number of partitions of n using only parts from T and not repeating any parts. Prove that $p_S(n) = p_{T,\text{dist}}(n)$ for all n.

(4) Use the notation from HW2 #3. Show that for $k \geq 0$, we have

$$\sum_{n \geq 0} \frac{c(n,k)}{n!} x^n = \frac{(-\mathbf{L}(1-x))^k}{k!}.$$

(5) Let V, W be $\mathbf{F}_q$-vector spaces with $\dim V = n$ and $\dim W = m$.
   (a) How many linear maps $V \to W$ are there?
   (b) Suppose $n \geq m$. How many surjective linear maps $V \to W$ are there?
   (c) Pick $k \leq \min(m,n)$. How many rank k linear maps $V \to W$ are there?

1. Optional problems (don't turn in)

(6) Let a_n be the number of ways to give n dollars using bills of size $1, 2, 5, 10, 20, 50$ such that at most three 20 dollar bills can be used. Give a simple formula for the generating function $A(x) = \sum_{n \geq 0} a_n x^n$.

(7) This one is challenging (I can't see a solution that is simple, but it doesn't require anything outside of this class). Prove:

$$\sum_{n \geq 1} x^{n(n-1)/2} = \prod_{n \geq 1} \frac{1 - x^{2n}}{1 - x^{2n-1}}.$$

(8) Let λ be an integer partition of n whose Durfee square has size $r \times r$. Call λ **almost self-conjugate** if $\lambda_i = \lambda_i^T + 1$ for $i = 1, \ldots, r$. Let $q(n)$ be the number of almost

self-conjugate partitions of n ($q(0) = 0$). Find a formula for

$$\sum_{n \geq 0} q(n) x^n$$

in the same spirit as Example 3.27.

(9) Consider a group of $n+1$ people of different ages and consider the following scenario:
- The youngest person becomes a zombie, and all others start as non-zombies.
- Each zombie may turn any non-zombie older than them into a zombie.
- Once a zombie, a person stays a zombie.
- Everyone eventually becomes a zombie.

At the end, we get a set of pairs $\{(i,j) \mid i \text{ infected } j\}$. Call this a **zombie set** of size $n+1$. For $n, k \geq 0$, show that $c(n,k)$ is the number of zombie sets of size $n+1$ such that the youngest person infects k people.

(10) Show that the number of complete flags in $\mathbf{F}_q^n$ is $[n]_q!$.

(11) (a) Let $r \geq 0$ be a non-negative integer. Show that $c(n+r, n)$ is a polynomial function of n of degree $2r$.
(b) Compute this polynomial for $r = 1, 2, 3, 4$.