Walks in graphs

Def. A *graph* G is (V, E), V = set (vertices)

E = multiset of $\binom{V}{2} \cup V$ (edges)

$\binom{V}{2}$ ← 2-element subsets of V

edges in V are called *loops*.

A *directed graph* is (V, E) where E is a multiset of $V \times V$

(v_1, v_2) means $v_1 \longrightarrow v_2$

Suppose $V = \{v_1, \dots v_n\}$, let a_{ij} = # of edges between v_i & v_j

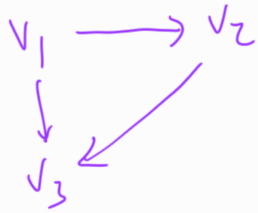
If V is directed, a_{ij} = # edges pointing from v_i to v_j.

A_G = $n \times n$ matrix $(A_G)_{ij} = a_{ij}$.

$G =$ 

$$A_G = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Note: in (undirected) case, A_G is symmetric

$H =$ 

$$A_H = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Def. A *walk* in a graph G is a sequence $v_0, e_1, v_1, \dots, e_k, v_k$

where $v_0, v_1, \dots v_k$ vertices, $e_1, \dots e_k$ are edges s.t:

undirected case: $e_i = \{v_{i-1}, v_i\}$

directed case: $e_i = (v_{i-1}, v_i)$

The beginning of walk is v_0, end is v_k. <u>length</u> is k

walk is <u>closed</u> if $v_0 = v_k$. (length 0 = choice of vertex)

<u>Thm</u>. Let G be (un)directed graph w/ vertices $v_1, \dots, v_n$. let $A = A_G$ be adjacency matrix. For all $k \geq 0$, the # of walks from v_i to v_j of length k is $(A^k)_{ij}$

($A^0 = \mathrm{id}$ by convention)

<u>Pf</u>. Induction on k. If $k=0$, clear.

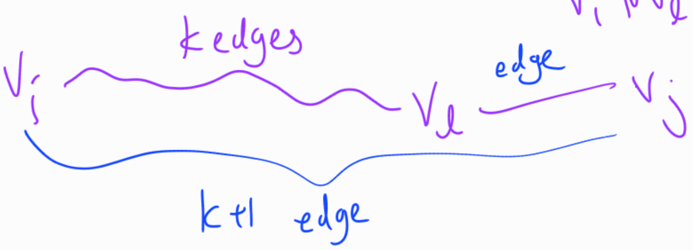
Now suppose result holds for k, let $B = A^k$.

$$(A^{k+1})_{ij} = (BA)_{ij} = \sum_{l=1}^{n} \underbrace{B_{il} A_{lj}}$$

edges from v_l to v_j

walks from v_i to v_l of length k

walks of length $k+1$ from v_i to v_j that stop at v_l right before v_j



$\Rightarrow (A^{k+1})_{ij} =$ # walks of length $k+1$ from v_i to v_j. $\square$

<u>Rmk</u>. Spectral thm: $n \times n$ symmetric matrix w/ real entries is always diagonalizable (and its eigenvalues are real).

A_G, if G undirected, is symmetric w/ real values (so diagonalizable)

If $A_G = BDB^{-1}$, D diagonal, then $A_G^k = BD^kB^{-1}$

Prop. $A = n \times n$ matrix w/ eigenvalues $\lambda_1, \dots, \lambda_n$. Then eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_n^k$. Hence:

$$\sum_{i=1}^{n} (A^k)_{i,i} = \operatorname{trace}(A^k) = \lambda_1^k + \dots + \lambda_n^k$$

In particular, if $A = A_G$, $\operatorname{trace}(A_G^k) = \lambda_1^k + \dots + \lambda_n^k$ is # of closed walks of length k in G.

Pf. Suppose A diagonalizable. Let $u_1, \dots u_n$ be eigenbasis for A w/ eigenvalues $\lambda_1 \dots \lambda_n$. Then $A^k u_i = \lambda_i^k u_i$

So $u_1, \dots u_n$ also eigenbasis for A^k, and $\lambda_1^k, \dots, \lambda_n^k$ are eigenvalues. $\square$

Ex. Fix integer $n \geq 1$. Let $f(k) = \#$ words $a_1 a_2 \cdots a_k$ in alphabet $[n]$ s.t. (1) $a_i \neq a_{i+1}$ for $i = 1, \dots k-1$ &

(2) $a_k \neq a_1$.

If we only impose (1), we get $n(n-1)^{k-1}$

Build graph G w/ vertices $v_1, \dots, v_n$ and an edge between v_i & v_j iff $i \neq j$. $f(k) = \#$ walks of length $k-1$ s.t. beginning vertex $\neq$ end vertex $= \sum_{i \neq j} (A_G^{k-1})_{ij}$

$$f(k) + \underbrace{\#\text{closed walks of length } k-1}_{\operatorname{trace}(A_G^{k-1})} = \#\text{ of walks of length } k-1 \text{ in } G = n(n-1)^{k-1}$$

Eigenvalues of $A_G = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & 0 & \ddots \\ & & & 0 \end{pmatrix}$

let $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$

Claim: $e_1 + \cdots + e_n$ $\quad n-1$ $\quad$ eigenvalues.

$\underbrace{e_1 - e_2}_{-1}, \underbrace{e_2 - e_3}_{-1}, \ldots, \underbrace{e_{n-1} - e_n}_{-1}$

are eigenbasis for A_G

Note: $A_G(e_i) = (e_1 + \cdots + e_n) - e_i$

$A_G(e_1 + \cdots e_n) = (n-1)(e_1 + \cdots + e_n)$, $\quad A_G(e_i - e_{i+1}) = e_{i+1} - e_i$

$\operatorname{trace}(A_G^{k-1}) = (n-1)^{k-1} + (n-1)(-1)^{k-1}$.

$$\Longrightarrow f(k) = n(n-1)^{k-1} - \left[(n-1)^{k-1} + (n-1)(-1)^{k-1}\right]$$

$$= (n-1)^k + (-1)^k (n-1)$$

$\square$